

ON THE FUNCTIONAL EQUATIONS INVOLVING NONLINEAR GENERALIZED P -COMPACT OPERATORS

BY

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Introduction. In his recent papers [23], [24], [25] Petryshyn has introduced and studied a class of *projectionally compact* (P -compact) nonlinear mappings of a real Banach space with property $(\pi)_K$ into itself. The main result of this study was an elementary and essentially a constructive proof of two fixed point theorems for bounded [23], [24] and unbounded [25] P -compact operators which were then used to derive a number of results concerning the existence and construction of solutions of various classes of nonlinear equations. Since, as was shown in [25], P -compact operators include, among others, completely continuous, quasicompact, and various classes of monotone operators (in a Hilbert space), the fixed point theorems (for balls about the origin) of Schauder [34], Rothe [32], Krasnoselsky [17], Altman [1] and Kaniel [14] as well as certain existence theorems for monotone operators of Zarantonello [41], Kacurovski [13], Minty [21], Browder [2], [3], Dolph and Minty [10] and others were deduced from the results on P -compact operators. Further results for P -compact operators or their extensions have been recently obtained by Lees and Schultz [18], De Figueiredo [8], [9] and Tucker [36].

The purpose of this paper is to continue the study and to extend the results for P -compact operators obtained in [23], [24], [25] to generalized P -compact or P_γ -compact operators introduced in [28]. One of our objectives is to show that many properties and results which are true for completely continuous operators carry over to generalized P -compact operators.

We now outline briefly the main results of this paper.

In §1 we introduce and discuss some basic definitions to be used in this paper.

In §2 we discuss further properties of P -compact and generalized P -compact operators. The main result of this section is Theorem 2.4 which asserts that under certain conditions “the Fréchet derivative $A'(x)$ is P -compact if and only if $A(x)$ is P -compact.”

In §3 we discuss certain properties of linear P -compact operators. The main result of this section is the characterization Theorem 3.3 which asserts that “a symmetric linear mapping A of a Hilbert space into itself is P -compact if and only if $A = S + T$ where $S \leq 0$ and T is completely continuous.”

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In §4 the fixed point theorems of [23], [24], [25] are extended to generalized P -compact operators to cover fixed point theorems for bounded closed convex sets with an interior. The main result of this section is Theorem 4.4 for unbounded P_γ -compact operators from which we then derive a number of known fixed point theorems mentioned above for convex sets as well as some recent results of Yamamuro [40], Hanani, Netanyahu and Reichaw-Reichbach [12] and others.

In §5 we apply our general fixed point theorem to the solution of various classes of nonlinear equations. In particular, we deduce the results of Krasnoselsky [16], [17], Ehrmann [11], certain results related to fixed point theorems of Browder [4] and others.

In §6 we first prove the convergence of an iteration method for the construction of fixed points of nonexpansive P_1 -compact mappings by utilizing the results and arguments of Browder-Petryshyn [6] and Krasnoselsky [16]; then we study a general (nonprojective) approximation method in the solution of functional equations. As special cases we deduce the corresponding results of Petryshyn [24] and Vainikko [39].

1. Basic definitions. In this paper it will always be assumed that X is a real Banach space such that there exists a pair of sequences $(\{X_n\}, \{P_n\})$, where each X_n is a finite dimensional subspace of X and P_n is a linear projection of X onto X_n such that $P_n x \rightarrow x$ as $n \rightarrow \infty$ for each x in X . The system $(\{X_n\}, \{P_n\})$ having the above property is said to be *projectionally complete* [27], [30]. In what follows we use " \rightarrow " and " \rightharpoonup " to denote *strong* and *weak* convergence in X , respectively, and H to denote a real Hilbert space.

Since some of the terms used in this paper are not universally accepted by people working in the field, to avoid possible confusion, we first state the following definitions. Let A be a (not necessarily linear) mapping of a (not necessarily linear) set $D(A) \subset X$ into X . A is said to be *precompact* if A maps every bounded set in $D(A)$ into a precompact set in X ; A is *completely continuous* if A is continuous and precompact; A is *strongly continuous* if $\{x_n\} \subset D(A)$, $x \in D(A)$, and $x_n \rightarrow x$ implies $Ax_n \rightarrow Ax$; A is *weakly continuous* if $\{x_n\} \subset D(A)$, $x \in D(A)$, and $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$; A is *hemicontinuous* if A is continuous from line segments in $D(A)$ to the weak topology in X ; A is *demiclosed* if $\{x_n\} \subset D(A)$, $x_n \rightarrow x$, $Ax_n \rightharpoonup y$ imply $x \in D(A)$ and $Ax = y$; A is *closed* if $\{x_n\} \subset D(A)$, $x_n \rightarrow x$, $Ax_n \rightarrow y$ imply $x \in D(A)$ and $Ax = y$; A is *strongly closed* if $\{x_n\} \subset D(A)$, $x_n \rightarrow x$, $Ax_n \rightarrow y$ imply $x \in D(A)$ and $Ax = y$; A is *bounded* if A maps bounded sets in $D(A)$ into bounded sets in X ; A is *monotone increasing* on $D(A) = X = H$ if $(Ax - Ay, x - y) \geq 0$ for all x and y in H ; A is *monotone decreasing* if $-A$ is monotone increasing. We also use $B_r(x_0)$ and $S_r(x_0) = \partial B_r(x_0)$ to denote the ball (i.e., $B_r(x_0) = \{x \in X \mid \|x - x_0\| \leq r\}$) and its boundary.

The class of operators to be studied in this paper is described by the following definition.

DEFINITION 1.1 [28]. A nonlinear operator A of $D(A)$ ($\subset X$) into X is called a *generalized P -compact* mapping or *P_γ -compact* if $P_n A$ is continuous in X_n for all sufficiently large n and if there exists a constant $\gamma \geq 0$ such that for any p dominating γ (i.e., $p \geq \gamma$ if $\gamma > 0$ and $p > \gamma$ if $\gamma = 0$) and any bounded sequence $\{x_n\}$ with $x_n \in X_n \cap D(A)$ the strong convergence of the sequence $\{P_n A x_n - p x_n\}$ implies the existence of a strongly convergent subsequence $\{x_{n_i}\}$ and an element $x \in D(A)$ such that $x_{n_i} \rightarrow x$ and $P_{n_i} A x_{n_i} \rightarrow Ax$ as $n_i \rightarrow \infty$.

REMARK 1.1. It was pointed out in [28] that the concept of P_γ -compactness is a generalization of the concept of *projectional compactness* or *P -compactness* introduced and studied in [23], [24], [25]. Indeed, when in Definition 1.1 we take $\gamma = 0$, $D(A) = X$ and $(\{X_n\}, \{P_n\})$ so that

$$(\pi)_K \quad P_n X = X_n; X_n \subset X_{n+1} \quad (n = 1, 2, \dots); \quad \bigcup_{n=1}^{\infty} X_n = X; \|P_n\| \leq K \quad (K \geq 1),$$

then it is easy to see that condition $(\pi)_K$ implies the projectional completeness of $(\{X_n\}, \{P_n\})$. Thus in this case, the concept of P_0 -compactness, according to Definition 1.1, is just the concept of P -compactness. In what follows we shall refer to P_0 -compactness as P -compactness although we no longer require that $(\{X_n\}, \{P_n\})$ satisfy condition $(\pi)_K$ or that $D(A)$ be the entire space X . We remark that upon examination of the proofs of the fixed point theorems in [23], [24], [25] it is clear that the only properties of A and X which were used are contained in the requirement that A be P_1 -compact according to Definition 1.1 as a mapping from $B_r(0)$ to X . It turns out that the class of operators defined by the generalized Definition 1.1 is not only larger but, at the same time, is more suitable for theoretical discussions and applicational purposes.

2. Properties and examples of nonlinear P_γ -compact operators. It was shown in [24] that the class of bounded P -compact operators A with $D(A) = X$ contains as its subclasses all closed precompact operators and, in particular, completely continuous operators (and strongly continuous operators if X is reflexive); all operators A such that $-A$ is quasicompact (see [14], [24]); and all bounded continuous, demicontinuous and weakly continuous monotone decreasing operators A defined on H . Consequently, the various fixed point theorems in [34], [32], [17], [1], [14] and existence theorems in [21], [2], [3] for monotone operators were deduced as special cases of the fixed point theorems for P -compact operators.

In this section we discuss further classes and properties of nonlinear (not necessarily bounded) P -compact operators. At the same time we obtain some new results for monotone operators in H .

THEOREM 2.1. *If A is a hemicontinuous monotone decreasing operator of $D(A) = H$ into H such that $P_n A P_n x \rightarrow Ax$ ($n \rightarrow \infty$) for each x in H , then A is P -compact.*

Proof. First note that since $P_n A$, as a mapping of the finite-dimensional space X_n into X_n , is hemicontinuous and monotone decreasing, it is also continuous [15].

Now let $\{x_n\}$ be any bounded sequence in X with $x_n \in X_n$ so that for some $p > 0$

$$g_n \equiv P_n A x_n - p x_n \rightarrow g \quad \text{in } X \text{ as } n \rightarrow \infty.$$

Since $-A$ is monotone increasing and $p > 0$, the operator $T = -A + pI$ is strongly monotone on H , i.e.,

$$p \|x - y\|^2 \leq (Tx - Ty, x - y) \quad \text{for all } x, y \in H,$$

and $P_n T x_n \rightarrow -g$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ and an element x_0 in X such that $x_m \rightharpoonup x_0$. Since $P_m x_m = x_m$,

$$\begin{aligned} p \|x_m - P_m x_0\| &\leq (Tx_m - TP_m x_0, x_m - P_m x_0) = (P_m T x_m - P_m TP_m x_0, x_m - P_m x_0) \\ &= (P_m T x_m, x_m) - (P_m TP_m x_0, x_m) - (P_m T x_m, P_m x_0) + (P_m TP_m x_0, P_m x_0). \end{aligned}$$

By assumption, $P_m T x_m \rightarrow -g$, $x_m \rightharpoonup x_0$ and $P_m A P_m x_0 \rightarrow A x_0$, so passing to the limit in the last inequality, we see that $\|x_m - P_m x_0\| \rightarrow 0$ as $m \rightarrow \infty$. This and the inequality

$$\|x_m - x_0\| \leq \|x_m - P_m x_0\| + \|P_m x_0 - x_0\|$$

imply that $x_m \rightarrow x_0$ as $m \rightarrow \infty$.

To show that $P_m A x_m \rightarrow A x_0$, we note first that for $\tilde{A} = -A$, $P_m \tilde{A} x_m = -g_m - p x_m \rightarrow -g - p x_0$ as $n \rightarrow \infty$, and that for any y in H ,

$$(P_m \tilde{A} P_m y - P_m \tilde{A} x_m, P_m y - x_m) = (\tilde{A} P_m y - \tilde{A} x_m, P_m y - x_m) \geq 0.$$

Thus on passage to the limit, we derive the inequality

$$(\tilde{A} y + g + p x_0, y - x_0) \geq 0 \quad \text{for all } y \text{ in } H.$$

If in the last inequality we take $y = x_0 + tz$ for any z in H and $t > 0$, then it reduces to

$$(\tilde{A}(x_0 + tz) + g + p x_0, tz) \geq 0$$

or, after cancelling the positive factor $t > 0$, to

$$(\tilde{A}(x_0 + tz) + g + p x_0, z) \geq 0.$$

Since A (and hence \tilde{A}) is hemicontinuous and $x_0 + tz \rightarrow x_0$ as $t \rightarrow 0^+$, we obtain the relation

$$(\tilde{A} x_0 + g + p x_0, z) \geq 0 \quad \text{for all } z \text{ in } H.$$

Similarly, taking $y = x_0 - tz$ with $t > 0$, we get

$$(\tilde{A} x_0 + g + p x_0, z) \leq 0 \quad \text{for all } z \text{ in } H.$$

The two inequalities imply that $A x_0 = g + p x_0$, so A is P -compact.

COROLLARY 2.1. *Every continuous monotone decreasing mapping A of H into H is P -compact.*

Proof. The assertion follows immediately from Theorem 2.1 since if A is continuous in H , then A is certainly hemicontinuous and $P_n A P_n x \rightarrow Ax$ for each x in H .

COROLLARY 2.2. *If A satisfies the conditions of Theorem 2.1 (and in particular of Corollary 2.1), then for any given $\mu > 0$ the mapping $A - \mu I$ is one-to-one and onto.*

Proof. This is an immediate consequence of Theorem 2.1 and Corollary 2 in [25].

In our discussion below we will need the following simple lemma.

LEMMA 2.1. *For the class of P -compact mappings A of X into X , the following assertions are valid:*

- (a) *The identity map $I: X \rightarrow X$ is not P -compact, but $-I$ is P -compact.*
- (b) *If A is P -compact then $-A$ is not necessarily P -compact, but $-A$ is P -compact with $p < 0$. (See [24] for the discussion of the case $p < 0$.)*
- (c) *If A is P -compact and $a \geq 0$, then aA is also P -compact.*
- (d) *If A and B are P -compact, then $A - B$ need not be P -compact.*
- (e) *If A is P -compact and f is a fixed element in X , then $A + f$ and $A - f$ are P -compact.*
- (f) *If A is P -compact and B is completely continuous, then the mapping $(1 - t)A + tB$ is P -compact for any fixed t in $[0, 1]$.*

Proof. The assertions (a)–(e) are obvious while the proof of (f) is essentially given in [25] and, therefore, will be omitted.

We now derive a number of results for the Lipschitzian type mappings which we shall use in the section on fixed point theorems.

THEOREM 2.2. *If A is a nonlinear mapping of $D(A) = X$ into X such that for some constant $L > 0$*

$$\|Ax - Ay\| \leq L\|x - y\| \quad \text{for all } x \text{ and } y \text{ in } X,$$

then A is P_γ -compact with any fixed $\gamma > KL$, where $K \geq 1$ is the uniform bound on the sequence $\{P_n\}$.

Proof. First, since A is Lipschitzian, $P_n A$ is continuous in X_n for each n . Next, since $P_n x \rightarrow x$ as $n \rightarrow \infty$ for each x in X , the uniform boundedness principle implies that there exists a constant $K \geq 1$ such that $\|P_n\| \leq K$ for all n . Suppose now that $\{x_n\}$ is an arbitrary bounded sequence in X with $x_n \in X_n$ so that for any fixed $\gamma > KL$ and any $p \geq \gamma$,

$$g_n = P_n A x_n - p x_n \rightarrow g \quad \text{in } X \text{ as } n \rightarrow \infty.$$

Consider the operators $D = pI - A$ and $D_n = pI - P_n A$ and note that for each n and all x and y in X_n

$$\|D_n x - D_n y\| \geq \alpha \|x - y\| \quad \text{with } \alpha = p - KL > 0.$$

Since $p - L \geq p - KL > 0$, it follows that the mapping $B = p^{-1}A$ is a strict contraction on X . Hence $I - B$ is a bicontinuous mapping of X onto X , by the usual iteration argument. Consequently, $D = pI - A$ is a one-to-one mapping of X onto X . Thus, Theorem 2 in [29] implies that the mapping D satisfies condition (H) which, as was noted in [29], is precisely the condition that A be P_γ -compact for $\gamma > KL$.

COROLLARY 2.3. *If A is a strict contraction on X (i.e., $L < 1$) and the sequence $\{P_n\}$ is such that $\|P_n\| = 1$ (i.e., $K = 1$), then A is P_γ -compact for any fixed $\gamma > L$.*

THEOREM 2.3. *If A is a nonexpansive mapping of X into X (i.e., $\|Ax - Ay\| \leq \|x - y\|$ for all x and y in X), then the mapping $(A - I)$ is P_γ -compact with $\gamma = K - 1$ if $K = 1$ and with any fixed $\gamma > K - 1$ if $K > 1$.*

Proof. It is clear that $P_n A$ is continuous in X_n for each n . Now suppose that $\{x_n\}$ is any bounded sequence in X with $x_n \in X_n$ so that for some p dominating γ , $g_n = P_n(A - I)x_n - px_n \rightarrow g$ as $n \rightarrow \infty$. Consider the operators $D = A - (1 + p)I$ and $D_n = P_n(A - (1 + p)I)$. Note that for all x and y in X_n and all n ,

$$\|D_n x - D_n y\| \geq \alpha \|x - y\| \quad \text{with } \alpha = 1 + p - K > 0.$$

Since, under our conditions on p , $D = A - (1 + p)I$ is a bicontinuous mapping of X onto X , the conclusion of this theorem follows from Theorem 2 in [29].

COROLLARY 2.4. *If $\|P_n\| = 1$ for each n , and A is a nonexpansive mapping of X into X , then $A - I$ is P -compact.*

REMARK 2.1. Using the same arguments as in [25], one shows that if A is P_γ -compact and B is completely continuous, then $A + B$ is P_γ -compact.

Let A be a nonlinear mapping of $D(A) = X$ into X . We recall that Ax is said to be *Fréchet-differentiable* at a given point x in X if there exists a continuous linear operator $A'(x)$ of X into X such that for any h in X ,

$$A(x + h) - Ax = A'(x)h + w(x; h) \quad \text{with } \frac{\|w(x; h)\|}{\|h\|} \rightarrow 0 \quad \text{as } \|h\| \rightarrow 0,$$

(i.e., to any given $\varepsilon > 0$, there exists a $\delta = \delta(x, h) > 0$ such that $\|w(x; h)\| \leq \varepsilon \|h\|$ if $\|h\| \leq \delta$). The operator $A'(x)$ is called the *Fréchet derivative* at x . Let X^* denote the conjugate space of X with the value of the linear functional e in X^* at x in X denoted by (x, e) .

It is known [38] that if A is completely continuous, then so is $A'(x)$ at any x , but whether the converse is true is not known⁽³⁾. If A is P -compact, we have the following result:

⁽³⁾ **Added in proof.** After this paper had been accepted for publication, the authors were informed by T. Tromba that R. A. Bonic, in his forthcoming paper to appear in J. Differential Geometry, has answered in negative the question posed by Vainberg [38, p. 51] by constructing examples of continuous mappings which are not completely continuous but whose derivatives are completely continuous.

THEOREM 2.4. *Suppose X is reflexive and the Fréchet derivative $A': X \rightarrow L(X, X)$ (*) is strongly continuous. Then at any given x in X , the operator $A'(x)$ is P -compact if and only if A is P -compact.*

Proof. Suppose that for any given x in X the operator $A'(x)$ is P -compact. Note first that since $A'(x)$ exists, by results in [38], A is continuous, and, therefore, $P_n A$ is continuous in X_n for each n . Suppose now that $\{x_n\} \subset X$ is an arbitrary bounded sequence with $x_n \in X_n$ such that for some $p > 0$, $g_n \equiv P_n A x_n - p x_n \rightarrow g$ as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded and X is reflexive, we may assume without loss of generality that x_n converges weakly to some element x_0 in X . For an arbitrary unit vector e_{nm} in X^* consider the equality

$$(\{P_n A x_n - p x_n\} - \{P_m A x_m - p x_m\}, e_{nm}) = (g_n - g_m, e_{nm})$$

whence, on adding and subtracting $P_n A x_0$ and $P_m A x_0$, we get

$$\begin{aligned} (\{P_n A x_n - P_n A x_0 - p x_n\} - \{P_m A x_m - P_m A x_0 - p x_m\}, e_{nm}) \\ = (g_n - g_m, e_{nm}) + (\{P_m A x_0 - P_n A x_0\}, e_{nm}). \end{aligned}$$

In view of the mean-value theorem for functionals, there exist numbers t_{nm} and t'_{nm} in $(0, 1)$ such that the last equality reduces to

$$\begin{aligned} (\{P_n A'(z_{nm})(x_n - x_0) - p x_n\} - \{P_m A'(z'_{nm})(x_m - x_0) - p x_m\}, e_{nm}) \\ = (g_n - g_m, e_{nm}) + (\{P_m A x_0 - P_n A x_0\}, e_{nm}), \end{aligned}$$

where $z_{nm} = x_0 + t_{nm}(x_n - x_0)$ and $z'_{nm} = x_0 + t'_{nm}(x_m - x_0)$, from which we derive the useful equality

$$\begin{aligned} (\{P_n A'(x_0)x_n - p x_n\} - \{P_m A'(x_0)x_m - p x_m\}, e_{nm}) \\ = (\{P_n A'(x_0) - P_n A'(z_{nm})\}(x_n - x_0), e_{nm}) + (g_n - g_m, e_{nm}) \\ - (\{P_m A'(x_0) - P_m A'(z'_{nm})\}(x_m - x_0), e_{nm}) + (P_m A x_0 - P_n A x_0, e_{nm}) \\ - (P_m A'(x_0)x_0 - P_n A'(x_0)x_0, e_{nm}). \end{aligned}$$

Now by the Hahn-Banach theorem we can choose e_{nm} so that $\|e_{nm}\| = 1$ and

$$\begin{aligned} (\{P_n A'(x_0)x_n - p x_n\} - \{P_m A'(x_0)x_m - p x_m\}, e_{nm}) \\ = \|\{P_n A'(x_0)x_n - p x_n\} - \{P_m A'(x_0)x_m - p x_m\}\| \equiv \|g'_n - g'_m\|. \end{aligned}$$

This and the preceding equality imply that

$$\begin{aligned} \|g'_n - g'_m\| &\leq \|P_n A'(x_0) - P_n A'(z_{nm})\| \|x_n - x_0\| + \|g_n - g_m\| \\ &\quad + \|P_m A'(x_0) - P_m A'(z'_{nm})\| \|x_m - x_0\| + \|P_m A x_0 - P_n A x_0\| \\ &\quad + \|P_m A'(x_0)x_0 - P_n A'(x_0)x_0\|. \end{aligned}$$

(*) We recall that $L(X, X)$ is the Banach space of all continuous linear operators of X into X with the topology in $L(X, X)$ being the metric topology defined by the bounds of the elements in $L(X, X)$. Then $A': X \rightarrow L(X, X)$ is strongly continuous means that $A'(z_n) \rightarrow A'(z)$ whenever $z_n \rightarrow z$ as $n \rightarrow \infty$.

Since $\{t_{nm}\}$ and $\{t'_{nm}\}$ belong to $(0, 1)$ and $x_n \rightarrow x_0$ and $x_m \rightarrow x_0$ it is clear that $z_{nm} \rightarrow x_0$ independent of m as $n \rightarrow \infty$ and $z'_{nm} \rightarrow x_0$ independent of n as $n, m \rightarrow \infty$. This, the boundedness of $\{x_n\}$, and the strong continuity of A' imply that the first and the third terms in the above inequality go to zero as $n, m \rightarrow \infty$. Furthermore, since $P_n u \rightarrow u$ for any u in X and $g_n \rightarrow g$ as $n \rightarrow \infty$, the other terms also go to zero as $n, m \rightarrow \infty$. Thus $\{g'_n\} = \{P_n A'(x_0)x_n - p x_n\}$ is a Cauchy sequence and therefore, by P compactness of $A'(x_0)$, there exists a subsequence $\{x_{n_i}\}$ and an element x in X such that $x_{n_i} \rightarrow x$. By the continuity of A , $P_{n_i} A x_{n_i} \rightarrow A x$. Consequently, A is P -compact.

Converse. Suppose now that A is P -compact and x is any given point in X . Suppose further that for any bounded sequence $\{x_n \mid x_n \in X_n\}$ and some $p > 0$

$$g_n = P_n A'(x)x_n - p x_n \rightarrow g \quad \text{as } n \rightarrow \infty.$$

First, by definition of the Fréchet derivative, $A'(x)$ is a bounded linear operator and, therefore, $P_n A'(x)$ is continuous in X_n for each n . Without loss of generality we may assume that $\{x_n\}$ converges weakly to some element x_0 in X . Now put $z_n = P_n x + x_n - P_n x_0$ and observe that $z_n \in X_n$ and $z_n \rightarrow x$ as $n \rightarrow \infty$. Choose a unit vector e_{nm} in X^* such that for $\tilde{g}_n = P_n A z_n - p z_n$ we have $\|\tilde{g}_n - \tilde{g}_m\| = (\tilde{g}_n - \tilde{g}_m, e_{nm})$. Thus adding and subtracting $P_n A x$ and $P_m A x$ we get

$$\begin{aligned} \|\tilde{g}_n - \tilde{g}_m\| &= ((P_n A z_n - P_n A x - p z_n) - (P_m A z_m - P_m A x - p z_m), e_{nm}) \\ &\quad + ((P_n A x - P_m A x), e_{nm}). \end{aligned}$$

As in the first part of the proof, by the mean-value theorem, there exists t_{nm} and t'_{nm} in $(0, 1)$ such that

$$\begin{aligned} \|\tilde{g}_n - \tilde{g}_m\| &= ((P_n A'(u_{nm})(z_n - x) - p z_n) - (P_m A'(u'_{nm})(z_m - x) - p z_m), e_{nm}) \\ &\quad + ((P_n A x - P_m A x), e_{nm}), \end{aligned}$$

where we have put $u_{nm} = x + t_{nm}(z_n - x)$ and $u'_{nm} = x + t'_{nm}(z_m - x)$. Now adding and subtracting $P_n A'(x)z_n$ and $P_m A'(x)z_m$ to the right-hand side of the above equality we obtain

$$\begin{aligned} \|\tilde{g}_n - \tilde{g}_m\| &= ((P_n A'(x)z_n - p z_n) - (P_m A'(x)z_m - p z_m), e_{nm}) \\ &\quad - ((P_n A'(x)x - P_m A'(x)x), e_{nm}) + ((P_n A'(u_{nm}) - P_n A'(x))(z_n - x), e_{nm}) \\ &\quad + ((P_m A'(x) - P_m A'(u'_{nm}))(z_m - x), e_{nm}) + ((P_n A x - P_m A x), e_{nm}). \end{aligned}$$

Since $z_n = P_n x + x_n - P_n x_0$ and $g_n = P_n A'(x)x_n - p x_n$ the last equality yields the looked-for relation

$$\begin{aligned} \|\tilde{g}_n - \tilde{g}_m\| &= (g_n - g_m, e_{nm}) + ((P_n A'(x)P_n x - P_m A'(x)P_m x), e_{nm}) \\ &\quad + ((P_m A'(x)P_m x_0 - P_n A'(x)P_n x_0), e_{nm}) + p(P_m x - P_n x + P_n x_0 - P_m x_0, e_{nm}) \\ &\quad + ((P_m A'(x)x - P_n A'(x)x), e_{nm}) + ((P_n A x - P_m A x), e_{nm}) \\ &\quad + ((P_n A'(u_{nm}) - P_n A'(x))(z_n - x), e_{nm}) + ((P_m A'(x) - P_m A'(u'_{nm}))(z_m - x), e_{nm}). \end{aligned}$$

Since $g_n \rightarrow g$, $P_n u \rightarrow u$ for each $u \in X$, $A'(x)$ is continuous and $A'(u_{nm}) \rightarrow A'(x)$ in $L(X, X)$, it follows from the above equality that $\{\tilde{g}_n\}$ is a Cauchy sequence. Thus, by the P -compactness of A , there exists a subsequence $\{z_{n_i}\} = \{P_{n_i}x + x_{n_i} - P_{n_i}x_0\}$ and an element y in X such that $z_{n_i} \rightarrow y$ and, therefore, $x_{n_i} = z_{n_i} + P_{n_i}x_0 - P_{n_i}x \rightarrow y + x_0 - x \equiv \tilde{x}$. By the continuity of $A'(x)$, $P_{n_i}A'(x)x_{n_i} \rightarrow A(x)\tilde{x}$, i.e., $A'(x)$ is P -compact.

3. Linear P -compact operators. In this section, we discuss the properties of linear P -compact mappings of X into X . The main result of this section is the characterization theorem which states that if X is a real Hilbert space H and A is a symmetric linear operator of H into H , then A is P -compact if and only if A is of the form $A = S + T$ where S is nonpositive and T is completely continuous in H . We start with the following theorem for linear (not necessarily bounded) operators acting in X .

THEOREM 3.1. (a) *Let A be a linear P -compact mapping of X into X and let $\lambda > 0$ be an arbitrary real number. Either the nonhomogeneous equation $\lambda x - Ax = y$ has a unique solution x in X whatever the given element y in X (and in particular has the unique solution $x = 0$ when $y = 0$) or the homogeneous equation $\lambda x - Ax = 0$ has nonzero solutions.*

(b) *If, additionally, we assume that A is bounded, then the number of linearly independent solutions of the homogeneous equation is finite.*

Proof. Suppose that the homogeneous equation $\lambda x - Ax = 0$ has $x = 0$ as its only solution. Then there exists an integer $N > 0$ and a constant $\alpha > 0$ such that

$$\|\lambda x - P_n Ax\| \geq \alpha \|x\| \quad \text{for all } x \text{ in } X_n \text{ and all } n \geq N.$$

If the latter assertion were not true for any $N > 0$, then we could find a sequence $\{x_n\}$ with $x_n \in X_n$ and, in view of the linearity of A , such that $\|x_n\| = 1$ and

$$\lambda x_n - P_n Ax_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But, by the P -compactness of A , this implies the existence of a subsequence $\{x_{n_i}\}$ and an element x in X such that $x_{n_i} \rightarrow x$ and $P_{n_i}Ax_{n_i} \rightarrow Ax$ as $n_i \rightarrow \infty$. Hence, $\|x\| = 1$ and $\lambda x - Ax = 0$ contradicting the fact that $\lambda x - Ax = 0$ has only a trivial solution. Thus, for $n \geq N$, $\lambda I - P_n A$ are one-to-one finite-dimensional linear operators and, therefore, are onto. Hence, for any given y in X , there exists a unique x_n in X_n such that $\lambda x_n - P_n Ax_n = P_n y$. Now, since $\alpha \|x_n\| \leq \|\lambda x_n - P_n Ax_n\| = \|P_n y\| \leq K \|y\|$ and $P_n y \rightarrow y$ as $n \rightarrow \infty$, it follows that $\{x_n\}$ with $x_n \in X_n$ is bounded. Thus, the P -compactness of A implies the existence of a subsequence $\{x_{n_i}\}$ and an element x in X such that $x_{n_i} \rightarrow x$ and $P_{n_i}Ax_{n_i} \rightarrow Ax$ as $n_i \rightarrow \infty$. This and our assumption imply that x is the unique solution of the nonhomogeneous equation.

To prove assertion (b), suppose now that A is bounded and that the null space, $N(\lambda I - A)$, contains nonzero elements. Then we claim that $N(\lambda I - A)$ is finite-dimensional. If this were not the case, then because the sphere $\|x\| = 1$ in the infinite

dimensional space $N(\lambda I - A)$ is not compact there exists a sequence of elements x_1, x_2, x_3, \dots ($\|x_k\| = 1$) in $N(\lambda I - A)$ such that $\|x_m - x_n\| > 1/2$ for $m \neq n$. Since the elements x of $N(\lambda I - A)$ are characterized by the relation $\lambda x - Ax = 0$ we have $\lambda x_k - Ax_k = 0$ for $k = 1, 2, \dots$. Now, since the system $(\{X_n\}, \{P_n\})$ is projectionally complete in X , for each k and $\varepsilon_k = 1/k$, there exists an integer $n(k)$ (which we can and shall assume that $n(k) > k$) such that $\|x_k - P_{n(k)}x_k\| < \varepsilon_k$. This, the boundedness of A and the equality $\lambda x_k - Ax_k = 0$ imply that with $w_{n(k)} = P_{n(k)}x_k$ in $X_{n(k)}$, we have

$$\begin{aligned} \|P_{n(k)}Aw_{n(k)} - \lambda w_{n(k)}\| &\leq \|P_{n(k)}Aw_{n(k)} - P_{n(k)}Ax_k\| + \|P_{n(k)}Ax_k - \lambda w_{n(k)}\| \\ &\leq K\|A\| \|w_{n(k)} - x_k\| + K\|Ax_k - \lambda x_k\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Thus, by the P -compactness of A , there exists a subsequence $\{w_{n(m)}\}$ of $\{w_{n(k)}\}$ and an element w in X such that $w_{n(m)} \rightarrow w$ and $P_{n(m)}Aw_{n(m)} \rightarrow Aw$ as $m \rightarrow \infty$. But $\|x_m - w\| \leq \|x_m - w_{n(m)}\| + \|w_{n(m)} - w\| \rightarrow 0$ as $m \rightarrow \infty$. Thus, $\{x_m\}$ is a convergent subsequence, contradicting the fact that $\|x_k - x_n\| > 1/2$ for $n \neq k$. Hence $N(\lambda I - A)$ is finite-dimensional.

COROLLARY 3.1 (LAX-MILGRAM LEMMA). *If A is a bounded linear operator of H into H such that $(Ax, x) \geq c\|x\|^2$ for all x in H and some constant $c > 0$, then A is one-to-one and onto.*

Proof. Since A is linear and $B = cI - A$ is such that $(Bx, x) = c(x, x) - (Ax, x) \leq 0$, Corollary 2.1 implies that B is P -compact. Let $\lambda > 0$ be any fixed real number and suppose that $\lambda x - Bx = 0$. Then $0 = ((\lambda I - B)x, x) = \lambda(x, x) - (Bx, x) \geq \lambda\|x\|^2$ implying that $x = 0$. Hence, by Theorem 3.1, the operator $(\lambda I - B)$ is one-to-one and onto for any given $\lambda > 0$. In particular, for $\lambda = c$ the operator $(cI - B) = A$ is one-to-one and onto.

Our next result is the following interesting theorem which plays an essential role in the characterization of symmetric P -compact operators.

THEOREM 3.2. *If A is a symmetric nonnegative (i.e., $A = A^*$ and $A \geq 0$) P -compact operator of H into H , then A is completely continuous.*

Proof. The proof of Theorem 3.2 ensues from the following lemmas.

LEMMA 3.1. *Under the assumptions of Theorem 3.2, the number $\mu = \|A\|$ is an eigenvalue of A , and the corresponding eigenspace H_μ is finite-dimensional.*

Proof. Consider the finite-dimensional operators $A_n = P_nAP_n$. They are symmetric and nonnegative as mappings of H_n into H_n . Hence for each n there exists an $x_n \in H_n$ with $\|x_n\| = 1$ such that

$$(3.1) \quad A_n x_n = \mu_n x_n, \quad \mu_n = \sup_{x \in H_n} \frac{\|A_n x\|}{\|x\|}.$$

It is obvious that for each n , $0 \leq \mu_n \leq \mu$ where $\mu = \sup_{x \in H} \|Ax\|/\|x\|$. Let $\bar{\mu}$ be the

l.u.b. of $\{\mu_n\}$. Hence there exists a subsequence $\{\mu_m\}$ of $\{\mu_n\}$ such that $\mu_m \rightarrow \bar{\mu}$ and $\bar{\mu} \leq \mu$. It follows from (3.1) that

$$(3.2) \quad A_m x_m - \bar{\mu} x_m = (\mu_m - \bar{\mu}) x_m \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Since A is P -compact, there exists a subsequence $\{x_{m_i}\}$ and an element \tilde{x} in H such that $x_{m_i} \rightarrow \tilde{x}$ and $A_{m_i} x_{m_i} \rightarrow A\tilde{x}$ as $m_i \rightarrow \infty$. Hence (3.2) implies that $A\tilde{x} - \bar{\mu}\tilde{x} = 0$, i.e., $\bar{\mu}$ is an eigenvalue of A and \tilde{x} is its corresponding eigenvector with $\|\tilde{x}\| = 1$. To prove that $\bar{\mu} = \mu$ note that $A_n x \rightarrow Ax$ for each x in H and that for each x in H and each n , with $P_n x = y_n$, we have

$$\|A_n y_n\| \leq \bar{\mu} \|y_n\| \quad \text{or} \quad \|A_n x\| \leq \bar{\mu} \|P_n x\|.$$

Passing to the limit we obtain $\|Ax\| \leq \bar{\mu} \|x\|$. This implies that $\bar{\mu} \geq \mu$ and hence $\bar{\mu} = \mu$.

The finite dimensionality of the eigenspace H_μ follows from assertion (b) of Theorem 3.1.

Now set $\bar{H} = H \ominus H_\mu$ and denote $\bar{A} \equiv A|_{\bar{H}}$. It is clear that $A: H_\mu \rightarrow H_\mu$, and since A is symmetric, it is also clear that $A: \bar{H} \rightarrow \bar{H}$. Thus, A is reduced by H_μ , and we have the standard result [35]:

LEMMA 3.2. (a) If \bar{P} and P_μ are projections of H onto \bar{H} and H_μ respectively, then H_μ reduces A if and only if $P_\mu A = AP_\mu$ and $\bar{P}A = A\bar{P}$.

(b) If H_μ reduces A , then it reduces $\lambda I - A$.

(c) If H_μ reduces A , and if \bar{A} and A_μ are restrictions of A to \bar{H} and H_μ respectively, then $R(A) = R(\bar{A}) \oplus R(A_\mu)$; A^{-1} exists if and only if \bar{A}^{-1} and A_μ^{-1} exist; $\sigma(A) = \sigma(\bar{A}) \cup \sigma(A_\mu)$; and $p\sigma(A) = p\sigma(\bar{A}) \cup p\sigma(A_\mu)$ where $p\sigma(A)$ denotes the point spectrum of A .

Thus, A can be represented as $A = \bar{A} \oplus A_\mu$, and any $x \in H$ has a unique representation as $x = \bar{x} + x_\mu$ with $\bar{x} \in \bar{H}$, $x_\mu \in H_\mu$. Furthermore, for any $x \in H$, $Ax = A\bar{x} + Ax_\mu = \bar{A}\bar{x} + A_\mu x_\mu$. We wish to prove that \bar{A} is a symmetric, nonnegative P -compact operator on \bar{H} . This will follow from the following general result.

LEMMA 3.3. If A is a symmetric nonnegative P -compact operator on H , $H = \bar{H} \oplus \tilde{H}$, where \tilde{H} reduces A , $\bar{A} = A|_{\bar{H}}$, then \bar{A} is a symmetric nonnegative P -compact operator on \bar{H} .

Proof. Symmetry and nonnegativeness are obvious. Let $(\{\bar{H}_n\}, \{\bar{P}_n\})$ and $(\{\tilde{H}_n\}, \{\tilde{P}_n\})$ be projectionally complete in \bar{H} and \tilde{H} , respectively. Suppose that $\{\bar{x}_n\}$ is a bounded sequence in \bar{H} with $\bar{x}_n \in \bar{H}_n$ so that for some $p > 0$,

$$\bar{P}_n \bar{A} \bar{x}_n - p \bar{x}_n \rightarrow \bar{g} \in \bar{H}.$$

Now put $H_n = \bar{H}_n \oplus \tilde{H}_n$ and $P_n = \bar{P}_n \oplus \tilde{P}_n$. It is clear that $(\{H_n\}, \{P_n\})$ is projectionally complete in H . Furthermore, let \tilde{x} be any element in \tilde{H} , and set $x_n = \bar{x}_n + \tilde{P}_n \tilde{x}$. Then $\{x_n\}$ is a bounded set in H with $x_n \in H_n$, and

$$\begin{aligned} P_n A x_n - p x_n &= P_n A (\bar{x}_n + \tilde{P}_n \tilde{x}) - p (\bar{x}_n + \tilde{P}_n \tilde{x}) \\ &= P_n A \bar{x}_n - p \bar{x}_n + P_n A \tilde{P}_n \tilde{x} - p \tilde{P}_n \tilde{x} \\ &= \bar{P}_n \bar{A} \bar{x}_n - p \bar{x}_n + P_n A \tilde{P}_n \tilde{x} - p \tilde{P}_n \tilde{x} \rightarrow \bar{g} + A \tilde{x} - p \tilde{x}. \end{aligned}$$

Since A is P -compact, there exists a subsequence $\{x_{n_i}\}$ with $x_{n_i} = \bar{x}_{n_i} + \tilde{P}_{n_i}\tilde{x}$ and an element $x \in H$ such that $x_{n_i} \rightarrow x$ and $P_{n_i}Ax_{n_i} \rightarrow Ax$. Therefore $\bar{x}_{n_i} \rightarrow x - \tilde{x} = \bar{x} \in \bar{H}$, and $\tilde{P}_{n_i}\bar{A}\bar{x}_{n_i} \rightarrow \bar{A}\bar{x}$, and hence, \bar{A} is a P -compact mapping of \bar{H} into \bar{H} .

Proof of Theorem 3.2. By Lemma 3.3, we can apply Lemma 3.1 to \bar{A} . Hence there exists an eigenvalue $\bar{\mu} < \mu$ of \bar{A} and a corresponding finite dimensional eigenspace $H\bar{\mu}$. Repeating the arguments above, this procedure furnishes a monotonically decreasing sequence of positive numbers

$$\mu = \mu_1, \quad \bar{\mu} = \mu_2, \dots, \mu_n, \dots,$$

where each μ_i appears as many times as its multiplicity. Thus, there exists a $\mu_0 \geq 0$ such that $\mu_i \rightarrow \mu_0$. We claim that $\mu_0 = 0$. For suppose $\mu_0 > 0$, and let $\{x_i\}$ be an orthonormal set of eigenvectors corresponding to $\{\mu_i\}$. Then constructing the set $\{w_{n(i)}\}$ as in Theorem 3.1, we have

$$\begin{aligned} \|P_{n(i)}Aw_{n(i)} - \mu_0 w_{n(i)}\| &\leq \|P_{n(i)}Aw_{n(i)} - P_{n(i)}Ax_i\| + \dots \\ &\quad + \|P_{n(i)}Ax_i - \mu_i P_{n(i)}x_i\| + \|\mu_i P_{n(i)}x_i - \mu_0 w_{n(i)}\| \\ &\leq \|A\| \|w_{n(i)} - x_i\| + \|Ax_i - \mu_i x_i\| + |\mu_i - \mu_0| \|x_i\| \rightarrow 0. \end{aligned}$$

Thus, as in the proof of Theorem 3.1, there exists a convergent subsequence $\{x_{i_j}\}$ which contradicts the orthonormality of the $\{x_i\}$. Hence $\mu_0 = 0$ and since, as is not hard to see, $Ax = \sum_{i=1}^{\infty} \mu_i(x, x_i)x_i$ for each x in H , it follows (see Riesz-Sz. Nagy [31]) that A is completely continuous.

The main result of this section is the following theorem:

THEOREM 3.3. *Let A be a symmetric linear mapping of H into H . Then A is P -compact if and only if A can be written in the form $A = S + T$, where $S \leq 0$ and T is completely continuous.*

Proof. Suppose $A = S + T$ where $S \leq 0$ and T is completely continuous. Then, by Corollary 2.1, S is P -compact and, by part (f) of Lemma 2.1, $A = S + T$ is P -compact.

Converse. Suppose A is a symmetric P -compact operator. Let E_0 be the orthogonal projection of H onto the eigenspace H_0 , the space corresponding to the nonpositive eigenvalues of A . It is clear that H_0 reduces A , and $A = AE_0 + A(I - E_0)$, where $AE_0 \leq 0$ and $A(I - E_0) \geq 0$. By Lemma 3.3, $T = A(I - E_0)$ is P -compact, so by Theorem 3.2, T is completely continuous. Thus, setting $S = AE_0$, $A = S + T$ with $S \leq 0$ and T completely continuous.

4. Fixed point theorems for generalized P -compact operators. The main purpose of this section is to generalize two major fixed point theorems established by Petryshyn for bounded P -compact operators A in [23], [24] and for unbounded P -compact operators A in [25]. Our extensions consist in showing that under the same conditions as in [24], [25] the corresponding fixed point theorems remain valid for the P_γ -mappings A which map an arbitrary *bounded, closed convex* set

$D(A)=C$ with an interior into X and which satisfy appropriate conditions on the boundary ∂C of C . Thus in our present case A need not be defined on the entire Banach space X . The proofs of the main results follow the arguments of Yamamuro [40] in the finite-dimensional case and those of [24], [25] in the infinite dimensional case. First, we shall prove the generalized theorems under the assumption that $0 \in \text{int } C$.

THEOREM 4.1. *Let A be a continuous mapping of a bounded, closed, convex set C of the finite dimensional Banach space X into X . Suppose that $0 \in \text{int } C$ and for some $\mu > 0$, A satisfies the condition:*

$(\pi_\mu^<)$: *If for some $x \in \partial C$ the equation $Ax = \alpha x$ holds then $\alpha < \mu$.*

Then there exists at least one element u in $(C - \partial C)$ such that $Au - \mu u = 0$.

Proof. For any $t \in [0, 1]$ consider the continuous homotopy mapping

$$A_t = t\{\mu I - A\} + \mu(1-t)I = \mu I - tA.$$

Then it is clear that $A_t x \neq 0$ for all x in ∂C and all $t \in [0, 1]$. For suppose there were a t_0 in $[0, 1]$ and an x_0 in ∂C such that $A_{t_0} x_0 = 0$. Then we would have $t_0 A x_0 = \mu x_0$. If $t_0 = 0$, then $x_0 = 0$, since $\mu > 0$, contradicting the hypothesis that $0 \in \text{int } C$. On the other hand, if $t_0 > 0$, then $A x_0 = (\mu/t_0)x_0$ with $\mu/t_0 \geq \mu$, contradicting the hypothesis that A satisfies the condition $(\pi_\mu^<)$ on ∂C . Thus, the degree⁽⁵⁾ of A_t at 0, as a mapping in C , is constant for $0 \leq t \leq 1$. Since the degree of $A_0 = \mu I$ is one, it follows that the degree of $A_1 = \mu I - A$ is also one. Therefore, there exists at least one element $u \in (C - \partial C)$ such that $A_1 u = \mu u - Au = 0$.

In view of the fact that Theorem 1 in [25] is an extension of Theorem 2 in [24], to generalize both theorems to the set C , it suffices to generalize Theorem 1.

THEOREM 4.2. *Suppose that A is a P_γ -compact mapping of a bounded closed convex set $D(A)=C$ of the Banach space X into X . Suppose further that $0 \in \text{int } C$ and that, for some μ dominating γ (i.e., $\mu \geq \gamma$ if $\gamma > 0$ and $\mu > \gamma$ if $\gamma = 0$), A satisfies both of the following conditions:*

(Λ) : *There exists a number $c > 0$ such that if for any n , $P_n A x = \lambda x$ holds for $x \in \partial C$ then $\lambda \leq c$.*

$(\pi_\mu^<)$: *If for some x in ∂C the equation $Ax = \alpha x$ holds then $\alpha < \mu$.*

Then there exists at least one element u in $(C - \partial C)$ such that $Au - \mu u = 0$.

Proof. The proof of Theorem 4.2 follows the same line of argument as that of Theorem 1 in [25]. All we need to show is that the present conditions (Λ) and $(\pi_\mu^<)$ imply the validity of the following lemma.

LEMMA 4.1. *If A satisfies the conditions of Theorem 4.2, then there exists an integer $n_0 > 0$ such that if $n \geq n_0$ and $P_n A x = \beta x$ for some x in $X_n \cap \partial C$, then $\beta < \mu$.*

⁽⁵⁾ For the definition and the properties of the "degree" of a mapping, see [19].

Proof of Lemma 4.1. If the assertion of Lemma 4.1 were not true for any n_0 , we could find a sequence $\{x_n\}$ with $x_n \in X_n \cap \partial C$ and a sequence of numbers $\{\beta_n\}$ such that

$$(4.1) \quad P_n A x_n = \beta_n x_n, \quad (\beta_n \geq \mu).$$

Hence our condition (Λ) implies that $\beta_n \leq c$, i.e., $\beta_n \in [\mu, c]$ for each n . Passing to a subsequence, we may assume that $\beta_n \rightarrow \beta$ and $\beta \in [\mu, c]$. This, (4.1), and the boundedness of C imply that

$$(4.2) \quad P_n A x_n - \beta x_n = (\beta_n - \beta) x_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since A is P_γ -compact and β dominates γ , (4.2) implies the existence of a strongly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and an element x in ∂C such that $x_{n_i} \rightarrow x$ and $P_{n_i} A x_{n_i} \rightarrow Ax$. This implies that $Ax - \beta x = 0$ for $x \in \partial C$ and $\beta \geq \mu$ in contradiction to condition $(\pi_\mu^<)$.

To complete the proof of Theorem 4.2, we see from Lemma 4.1 that there exists an integer $n_1 > 0$ such that for all $n \geq n_1$, $P_n A$ is continuous in $X_n \cap C$ and satisfies the condition $(\pi_\mu^<)$ on $X_n \cap \partial C$. Since $0 \in \text{int } C \cap X_n$ for all n , Theorem 4.1 implies the existence of at least one element u_n in $(C - \partial C) \cap X_n$ such that $P_n A u_n - \mu u_n = 0$. By P_γ -compactness of A , there exists a subsequence $\{u_{n_i}\}$ and an element u in C such that $u_{n_i} \rightarrow u$ and $P_{n_i} A u_{n_i} \rightarrow Au$ as $n_i \rightarrow \infty$ and $Au - \mu u = 0$. Note that the validity of the last equation implies that $u \in (C - \partial C)$, for the assumption that $u \in \partial C$ would lead to a contradiction of the condition $(\pi_\mu^<)$.

REMARK 4.1. The condition (Λ) of Theorem 4.2 is, for example, satisfied when A is either *bounded* or $A(\partial C) \subset C$. Indeed if A is bounded, then $\|P_n A x\| \leq M$ for some $M > 0$ and all x in ∂C ; therefore, if $P_n A x = \lambda x$ for any n and x in ∂C , then since the conditions $x \in \partial C$ and $0 \in \text{int } C$ imply that $\|x\| \geq \delta$ for some $\delta > 0$, it follows that $\lambda \leq M/\delta$ because $\delta|\lambda| \leq |\lambda| \|x\| = \|P_n A x\| \leq M$. On the other hand, if $A(\partial C) \subset C$, then denoting the diameter of C by $d(C) = \sup_{x, y \in C} \|x - y\|$ we see that, since $Ax \in C$ for any $x \in \partial C$, $\delta|\lambda| \leq \|Ax\| = \|P_n A x\| \leq K \|Ax\| \leq K d(C)$ from which our assertion follows.

An immediate consequence of Theorem 4.2 is the following general result asserting the existence of fixed points.

THEOREM 4.3. *Suppose that A is a P_γ -compact mapping of C (with $0 \in \text{int } C$) into X for $\gamma \leq 1$ such that A satisfies conditions (Λ) and $(\pi_1^<)$ on ∂C . Then A has at least one fixed point in $(C - \partial C)$.*

COROLLARY 4.1. *Suppose A is a P_γ -compact mapping of C ($0 \in \text{int } C$) into X for $\gamma \leq 1$. Then A has a fixed point in C provided*

- (1) $A(\partial C) \subset C$ (and, in particular, $A(C) \subset C$) or
- (2) A is bounded and $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$ for all x in ∂C .

Proof. In view of Theorem 4.3 and Remark 4.1, to prove Corollary 4.1 it suffices to show that (1), as well as (2) implies condition $(\pi_1^<)$ on ∂C .

First, we assume without loss of generality that A has no fixed points on ∂C . Suppose now that $A(\partial C) \subset C$ and $Ax = \alpha x$ for some x in ∂C . Since we want to show that $\alpha < 1$ we may assume that $\alpha \geq 0$. Hence, since by assumption $Ax \in C$, and $Ax = \alpha x$, it follows that $\alpha < 1$, i.e., (1) implies $(\pi_1^<)$.

To prove that (2) implies $(\pi_1^<)$, suppose $Ax = \alpha x$ for some x in ∂C . Then by (2), $(\alpha - 1)^2 \geq \alpha^2 - 1$ or $\alpha < 1$.

In the next theorem, we drop the condition that $0 \in \text{int } C$ but assume instead that C has an interior and employ Theorem 4.2 in the proof of the following main fixed point theorem of this paper.

THEOREM 4.4. *Suppose that A is a P_γ -compact mapping of a bounded closed convex set C (with an interior) into X . Suppose further that for some μ dominating γ the operator A satisfies both of the following conditions:*

($\tilde{\Lambda}$): *There exists a constant $c > 0$ such that if for some $x_0 \in \text{int } C$ and any n the equation $P_n A y - P_n x_0 = \lambda(y - x_0)$ holds for y in ∂C , then $\lambda \leq c$.*

($\tilde{\pi}_\mu^<$): *If for some x in ∂C the equation $Ax - x_0 = \alpha(x - x_0)$ holds, then $\alpha < \mu$.*

Then there exists at least one element u in $(C - \partial C)$ such that $Au - x_0 = \mu(u - x_0)$.

Proof. Consider the set $C' = C - x_0 = \{x - x_0 \mid x \in C\}$. It is easy to see that C' is a bounded closed convex set in X with $0 \in \text{int } C'$ and $\partial C' = \partial C - x_0$. Now define the mapping $A'(y')$ for y' in C' and $y' = \{x - x_0 \mid x \in C\}$ by $A'(y') = Ax - x_0$. Then A' maps C' into X and A' is P_γ -compact. Furthermore, if $P_n A' y' = \lambda y'$ for y' in $\partial C'$, then $\lambda \leq c$ by ($\tilde{\Lambda}$) since $P_n A' y' = P_n Ax - P_n x_0 = \lambda(x - x_0)$. Also, if $A'(y') = \alpha y'$ for some $y' \in \partial C'$, then by ($\tilde{\pi}_\mu^<$), $\alpha < \mu$ since $A'(y') = Ax - x_0 = \alpha(x - x_0)$. Thus, A' and C' satisfy all the conditions of Theorem 4.2. Hence there exists an element $u' \in (C' - \partial C')$ such that $A'(u') - \mu u' = 0$, or $Au - x_0 = \mu(u - x_0)$ with $u \in (C - \partial C)$.

THEOREM 4.5. *If A is a P_γ -compact operator for $\gamma \leq 1$ satisfying conditions ($\tilde{\Lambda}$) and ($\tilde{\pi}_1^<$) on ∂C of Theorem 4.4, then A has a fixed point in $(C - \partial C)$.*

REMARK 4.2. In case A is completely continuous, Theorem 4.5 was proved by Yamamuro [40] by employing the theory of the degree of a mapping.

A number of known fixed point theorems are deducible from the following corollary to Theorem 4.5.

COROLLARY 4.2. *If A is a P_γ -compact mapping of C (with an interior) into X for $\gamma \leq 1$, then A has a fixed point in C provided*

- (1) $A(\partial C) \subset C$ (and, in particular, $A(C) \subset C$) or
- (2) A is bounded and for some x_0 in $\text{int } C$,

$$\|Ax - x\|^2 \geq \|Ax - x_0\|^2 - \|x - x_0\|^2 \quad \text{for all } x \text{ in } \partial C.$$

Proof. In virtue of Theorem 4.5 it suffices to show that each of our conditions (1) and (2) implies conditions ($\tilde{\Lambda}$) and ($\tilde{\pi}_1^<$) on ∂C .

Without loss of generality we may assume that A has no fixed points on ∂C . If, as in the proof of Theorem 4.4, we take $C' = C - x_0 = \{x - x_0 \mid x \in C\}$, $\partial C' = \partial C - x_0$, and define $A'(y')$ for y' in C' and $y' = \{x - x_0 \mid x \in C\}$ by $A'(y') = Ax - x_0$,

then we see that $0 \in \text{int } C'$, A' maps C' into C' , A' has no fixed points on $\partial C'$ and, for all y' in $\partial C'$, $A'(y') \in C'$ in case (1) while $\|A'(y') - y'\|^2 \geq \|A'(y')\|^2 - \|y'\|^2$ in case (2). Thus, Corollary 4.2 follows from Corollary 4.1.

SPECIAL FIXED POINT THEOREMS. To indicate the generality of our Theorems 4.4 and 4.5 we show here that for Banach spaces having projectionally complete systems $(\{X_n\}, \{P_n\})$ many of the known fixed point theorems follow as special cases of our Theorem 4.5. Thus, our results supply elementary and essentially constructive proofs of these theorems.

THEOREM O. *Let A be a completely continuous mapping of a bounded closed convex set C (with an interior) into X . Then A has a fixed point in C provided A satisfies any one of the following conditions:*

- (S) (Schauder [34]): $A(C) \subset C$.
- (R) (Rothe [32]): $A(\partial C) \subset C$ in case $C = B_r(0)$.
- (A) (Altman [1]): $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$ for all x in $\partial C = \partial B_r(0)$.
- (T) (Tychonoff [37]): $Ax - \lambda x \neq 0$ for any x in ∂C and any $\lambda > 1$.

Proof. The proofs of the above theorems follow from Theorem 3 in [24] and Theorem 4.5 above. Let us add that if $X = H$, condition (A) is equivalent to the condition $\langle Ax, x \rangle \leq \|x\|^2$ for all x in $\partial B_r(0)$ first used by Krasnoselsky [17].

THEOREM Y (YAMAMURO). *Let A be a completely continuous mapping of a bounded closed convex set C (with an interior) into X . The mapping A has a fixed point in C provided that if for some x_0 in $\text{int } C$ the equation $Ax - x_0 = \alpha(x - x_0)$ holds for some x in ∂C then $\alpha < 1$.*

Proof. Theorem Y follows from Theorem 3 in [24] and Theorem 4.5 above.

THEOREM HNRR (HANANI, NETANYAHU, REICHAW-REICHBACH [12]). *Let A be a completely continuous mapping of $B_r(x_0)$ into H . Then A has a fixed point in $B_r(x_0)$ provided $\langle x - x_0, Ax - x_0 \rangle \leq r^2$ for all x in $\partial B_r(x_0)$.*

Proof. If $X = H$ and $C = B_r(x_0)$, then it is not hard to see that condition (2) of Corollary 4.2 is equivalent in this case to the condition $\langle x - x_0, Ax - x_0 \rangle \leq r^2$ for all x in $\partial B_r(x_0)$. Since a completely continuous operator is bounded and P -compact, Theorem HNRR follows from Corollary 4.2.

THEOREM A (PETRYSHYN). *If A is a bounded P -compact mapping of $B_r(0)$ into X such that A satisfies $(\pi_\mu^<)$ on $\partial B_r(0)$ for some $\mu > 0$, then there exists a point u in $(B_r(0) - \partial B_r(0))$ such that $Au - \mu u = 0$.*

REMARK 4.3. As was shown in [24], a special case of Theorem A is a fixed point theorem for quasicompact mappings established by Kaniel [14].

THEOREM B (PETRYSHYN). *If A is a P -compact mapping of $B_r(0)$ into X such that A satisfies condition (Λ) and $(\pi_\mu^<)$ on ∂B_r for some $\mu > 0$, then there exists a point u in $(B_r - \partial B_r)$ such that $Au - \mu u = 0$.*

Proof. Theorem B, which is an extension of Theorem A, is precisely Theorem 4.4 if we take $C = B_r(0)$ and $\gamma = 0$.

5. Applications of general fixed point theory to the solution of various equations.

In this section we apply the fixed point theorems obtained in the previous section to the solution of various classes of nonlinear equations.

THEOREM 5.1. Suppose $\|P_n\| = 1$ for all n and $A = S + T$ maps $D(A) = X$ into X , where T is completely continuous and S is strictly contractive with Lipschitz constant $L < 1$. If on the boundary ∂C of some bounded closed convex set C with an interior the mapping A satisfies condition $(\pi_1^<)$, then A has a fixed point in $(C - \partial C)$.

Proof. By Corollary 2.3, the mapping S is P_γ -compact for any fixed $\gamma > L$, while by Remark 2.1, the map $A = S + T$ is also P_γ -compact. Since $L < 1$ and A satisfies $(\pi_1^<)$ on ∂C , the result follows from Theorem 4.5 and Remark 4.1.

REMARK 5.1. In [16] Krasnoselsky showed that if S is a strict contraction on C , T is completely continuous and $A = S + T$ is such that

$$(K) \quad Sx + Ty \in C \quad \text{for all } x, y \in C,$$

then A has a fixed point in C . We see that, on the one hand, our Theorem 5.1 is weaker than Krasnoselsky's result since unlike him, we impose the conditions on the entire Banach space X while, on the other hand, Theorem 5.1 is stronger than his theorem since our condition $(\pi_1^<)$ on the boundary ∂C is much weaker and more convenient for applications than the rather strong and restrictive condition (K).

In case $X = H$ or $L < 1/2$ for $X \neq H$, then we have the following theorem which is stronger than Theorem 2 in [16]. Without loss of generality we assume that $0 \in \text{int } C$.

THEOREM 5.2. Suppose that $A = S + T$ is defined only on $D(A) = C \subset X$ with $0 \in \text{int } C$, where T is completely continuous and S is such that for all x and y in C

$$(5.1) \quad \|Sx - Sy\| \leq L\|x - y\|, \quad L < 1.$$

(a) If $X = H$ and A satisfies $(\pi_1^<)$ on ∂C , then A has a fixed point in $(C - \partial C)$.

(b) If $C = B_r(0) \subset X$, and (5.1) holds with $L < 1/2$ and $\|P_n\| = 1$ for all n , then A has a fixed point in C provided that A satisfies the condition $(\pi_1^<)$ on $\partial C = \partial B_r$.

Proof (a). Let R_C be the retraction of H onto C . Then it is not hard to show (see, for example, [7]) that R_C is nonexpansive. Thus, if we define the mapping \tilde{A} of H into H by $\tilde{A}x = SR_Cx + TR_Cx$ for each x in H , then $\tilde{A}x = Ax$ for each x in C , SR_C is strictly contractive on H , TR_C is completely continuous on H and \tilde{A} satisfies $(\pi_1^<)$ on ∂C . Hence, by Theorem 5.1, there exists a point x in $(C - \partial C)$ such that $\tilde{A}x = Ax = x$, i.e., A has a fixed point in $(C - \partial C)$.

(b) Suppose now that R_C is the retraction of X onto $C = B_r(0)$. Then R_C is given by

$$(5.2) \quad \begin{aligned} R_Cx &= x && \text{if } \|x\| \leq r, \\ &= \frac{rx}{\|x\|} && \text{if } \|x\| \geq r. \end{aligned}$$

The proof of this part depends on the following simple lemma:

LEMMA 5.1. *The retraction R_C of X onto $B_r(0)$ given by (5.2) is such that*

$$\|R_C x - R_C y\| \leq 2\|x - y\| \quad \text{for all } x \text{ and } y \text{ in } X.$$

Proof of Lemma 5.1. Lemma 5.1 is certainly true for all x and y in $B_r(0)$ while for x and y outside $B_r(0)$, i.e., for $\|x\| > r$ and $\|y\| > r$, the inequality

$$(5.3) \quad \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2}{\max(\|x\|, \|y\|)} \|x - y\|$$

established in [20] implies that

$$\|R_C x - R_C y\| = r \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq r \cdot \frac{2}{r} \|x - y\| = 2\|x - y\|.$$

Suppose now that $x \in B_r(0)$, $y \notin B_r(0)$. Then, since $\|y\| > r$,

$$\begin{aligned} \|R_C x - R_C y\| &= \left\| x - \frac{ry}{\|y\|} \right\| \leq \|x - y\| + \left\| y - \frac{ry}{\|y\|} \right\| \\ &\leq \|x - y\| + \left\{ 1 - \frac{r}{\|y\|} \right\} \|y\| = \|x - y\| + \|y\| - r \\ &\leq \|x - y\| + \|y\| - \|x\| \leq 2\|x - y\|. \end{aligned}$$

Thus we have shown that $\|R_C x - R_C y\| \leq 2\|x - y\|$ for all x and y in X .

To complete the proof of part (b) of Theorem 5.2 consider now the mapping \tilde{A} of X into X defined by $\tilde{A}x = SR_C x + TR_C x$ for all x in X . Our conditions imply that TR_C is completely continuous and SR is strictly contractive on all of X . Consequently, by Corollary 2.3 and Remark 2.1, the mapping \tilde{A} is P_γ -compact for any fixed $\gamma > 2L$. Furthermore, since $\tilde{A}x = Ax$ for all x in B_r , \tilde{A} satisfies condition $(\pi_1^<)$ on ∂B_r . Hence since $2L < 1$, by Theorem 5.1, \tilde{A} has a fixed point in $(B_r - \partial B_r)$, and, thus, A has a fixed point in $(B_r - \partial B_r)$.

THEOREM 5.3. *Let A be a bounded P -compact mapping of X into X such that*

$$(5.4) \quad \lim_{\|x\| \rightarrow \infty} \frac{\|Ax\|}{\|x\|} = 0.$$

Then for any given $\lambda > 0$ and any given y in X there exists x in X such that

$$(5.5) \quad x - \lambda Ax = y.$$

Proof. For an arbitrary but given $\lambda > 0$ and y in X consider the mapping $Tx = \lambda Ax + y$. Clearly T is a bounded P -compact operator. Furthermore, there exists a sphere S_r such that T satisfies condition $(\pi_1^<)$ on S_r . To determine S_r suppose that $\alpha x = Tx = \lambda Ax + y$ for some x in S_r . Then, assuming without loss of generality that $\alpha \geq 0$, we have

$$\alpha \|x\| \leq \lambda \|Ax\| + \|y\| \quad \text{or} \quad \alpha \leq \lambda \frac{\|Ax\|}{\|x\|} + \frac{\|y\|}{\|x\|}.$$

Since y is fixed, we can choose r_0 so that $\|y\|/\|x\| < \frac{1}{2}$ for $\|x\| \geq r_0$ while, since A satisfies (5.4) and λ is fixed, we can choose $r_1 > 0$ so that $\lambda\|Ax\|/\|x\| < \frac{1}{2}$ for $\|x\| \geq r_1$. Thus, for x in S_r with $r \geq \max\{r_0, r_1\}$, $\alpha < 1$. Hence by Theorem A, T has a fixed point in $(B_r - S_r)$, i.e., equation (5.5) is solvable.

THEOREM 5.4. *Let B be a positive symmetric mapping of H into H , $\lambda > 0$ and F a nonlinear mapping of H into H such that*

$$(\lambda Fy, y) \leq \lambda a_1(y, y) + a_2(y, y)^\eta \quad (a_2 > 0, 0 < \eta < 1) \quad \text{and} \quad \lambda a_1\|B\| < 1 \quad \text{if } a_1 > 0.$$

If AFA is a P -compact operator with $A^2 = B$, then the equation $x = \lambda BFx$ is solvable.

Proof. Since A is symmetric, for every x in H ,

$$(\lambda AFAx, x) = (\lambda FAx, Ax) \leq \lambda a_1\|B\| \|x\|^2 + a_2\|B\|^\eta \|x\|^{2\eta}.$$

It is easy to see that, since $\lambda a_1\|B\| < 1$ if $a_1 > 0$ and $1 - \eta > 0$, there exists a sphere S_r such that $\lambda a_1\|B\| r^2 + a_2\|B\|^\eta r^{2\eta} < r^2$ for $\|x\| = r$. Indeed, any fixed r satisfying the inequality

$$(5.6) \quad r > \left\{ \frac{a_2\|B\|^\eta}{1 - \lambda a_1\|B\|} \right\}^{1/(2-2\eta)}$$

has this property. Hence, for any fixed r satisfying (5.6),

$$(\lambda AFAx, x) \leq \|x\|^2 \quad \text{for all } x \text{ in } S_r.$$

Thus, as was noted in [24], the mapping AFA satisfies conditions (Λ) and $(\pi_1^<)$ on S_r . Since $\lambda > 0$ and AFA is P -compact, it follows that λAFA is P -compact. Hence, by Theorem B, there exists $x \in (B_r - S_r)$ such that $x = \lambda AFAx$ or $z = \lambda BFz$ with $z = Ax$.

REMARK 5.2. In virtue of Theorem 3 in [24] and Corollary 2.1, AFA is a P -compact mapping of H into H if, for example, F is a mapping of H into H which is either a continuous or a weakly continuous monotone decreasing mapping, a bounded demicontinuous monotone decreasing mapping or a completely continuous mapping. For the last class of mappings with $\lambda = 1$, Theorem 5.4 was proved in [17].

Our next theorem contains conditions which should prove convenient and useful for applications.

THEOREM 5.5. *Suppose that A is a P -compact mapping of $D(A)$ into X having the following properties:*

(1) *There exists $x_0 \in D(A)$ and numbers $r > 0$ and $c > 0$ so that $B_r(x_0) \subset D(A)$ and*

$$\|Ay - x_0\| \leq c \quad \text{for } y \in S_r(x_0) = \partial B_r(0).$$

(2) *If $r < c$, then in addition there exists a number $q < 1$ so that for $r_1 = (c - qr)/(1 - q)$, $B_{r_1}(x_0) \subset D(A)$ and $\|Ax - Ay\| \leq q\|x - y\|$ for each $x \in S_{r_1}(0)$ and $y = \alpha x_0 + (1 - \alpha)x \in S_r(x_0)$ for some $\alpha \geq 0$.*

Then A has a fixed point in $B_{r^}(x_0)$, where $r^* = r$ if $c \leq r$ and $r^* = r_1$ if $c > r$.*

Proof. If $c \leq r$, then $A(S_r(x_0)) \subset B_r(x_0)$ and, therefore, our conclusion follows from part (1) of Corollary 4.2.

Suppose now that $c > r$ and x is any given element in $S_{r_1}(x_0)$, where clearly $r < c < r_1$. Then setting $y = \alpha x_0 + (1 - \alpha)x$ we see that $y \in S_r(x_0)$ for $\alpha = 1 - r/r_1$. Hence, it follows from (1) and (2) that $\|Ax - x_0\| \leq \|Ax - Ay\| + \|Ay - x_0\| \leq q\|x - y\| + c$. Since $\|x - y\| = \alpha\|x_0 - x\| = (1 - r/r_1)r_1 = r_1 - r$, the last inequality and (2) imply that $\|Ax - x_0\| \leq q(r_1 - r) + c = r_1$, i.e., $A(S_{r_1}(x_0)) \subset B_{r_1}(x_0)$. Again, by part (1) of Corollary 4.2, A has a fixed point in $B_{r_1}(x_0)$.

REMARK 5.3. The conditions of Theorem 5.5 are certainly satisfied if in (1) we assume that $\|Ay - x_0\| \leq c$ for all y in the ball $B_r(x_0)$ while, in case $r < c$, the additional condition (2) is assumed to hold for all x and y in the shell $r \leq \|x - x_0\| \leq r_1$. Under these stronger assumptions (1) and (2), Theorem 5.5 was proved in [11] for the case when A is completely continuous. Note that our conditions (1) and (2) are not only weaker than the *ball* and *shell* conditions but are more suitable for applications and easier to verify.

We also remark that the application of Theorem 5.5 is easiest when the P -compact operator A is such that the condition (2) with $q < 1$ holds for all x and y outside a certain ball $B_r(x_0)$.

Theorem 5.5 can be applied to equations of the form

$$\begin{aligned} Ax &= (I - T)^{-1}B(I - T)x \quad \text{for } x \in D(B) = X \quad \text{or} \\ A_1x &= (I - T)^{-1}(B - T)x \quad \text{for } x \in D(B) \end{aligned}$$

provided that B is P -compact and T is a bounded linear operator such that $(I - T)^{-1}$ exists and is a bounded linear operator defined on all of X such that each subspace X_n reduces $(I - T)^{-1}$, i.e., $P_n(I - T)^{-1} = (I - T)^{-1}P_n$ for each n . The last condition would be satisfied, for example, if T were a completely continuous operator with a complete set of eigenelements $\{\phi_i\}$ ($i = 1, 2, \dots$) and $X_n = \text{span}\{\varphi_1, \dots, \varphi_n\}$ for each n .

LEMMA 5.2. *Under the above conditions on B and T*

(a) *The operator $A = (I - T)^{-1}B(I - T)$ is P -compact.*

(b) *The operator $A_1 = (I - T)^{-1}(B - T)$ is P -compact if T is completely continuous.*

Proof. It is obvious that the operators P_nA and P_nA_1 are continuous in X_n for n greater than some n_0 .

(a) Suppose now that $\{x_n\} \subset X$ is an arbitrary bounded sequence with $x_n \in X_n$ so that for some $p > 0$

$$(5.7) \quad g_n = P_nAx_n - px_n \rightarrow g \quad \text{as } n \rightarrow \infty.$$

Since $(I - T)$ is continuous and $(I - T)^{-1}$ is reduced by X_n for each n , it follows from (5.7) and the structure of A that

$$(5.8) \quad P_nB(I - T)x_n - pP_n(I - T)x_n = (I - T)g_n \rightarrow (I - T)g = h.$$

Here we have used the fact (see [35]) that if a linear operator is reduced by a subspace, then so is its inverse if it exists. Setting $y_n = (I - T)x_n$, we have that $y_n \in X_n$, $\{y_n\}$ is a bounded set and $P_n B y_n - p y_n \rightarrow h$. This, and the P -compactness of B imply that there exists a subsequence $\{y_{n_i}\}$ and an element y in X such that $y_{n_i} \rightarrow y$ and $P_{n_i} B y_{n_i} \rightarrow B y$. Since $(I - T)^{-1}$ is continuous, $x_{n_i} = (I - T)^{-1} y_{n_i} \rightarrow (I - T)^{-1} y = x$. Furthermore,

$$\begin{aligned} P_{n_i} A x_{n_i} &= P_{n_i} (I - T)^{-1} B (I - T) x_{n_i} = (I - T)^{-1} P_{n_i} B y_{n_i} \\ &\rightarrow (I - T)^{-1} B y = (I - T)^{-1} B (I - T) x = A x. \end{aligned}$$

Thus, A is P -compact.

(b) With $\{x_n\}$, $p > 0$ as above, suppose T is completely continuous and

$$P_n A_1 x_n - p x_n = g_n \rightarrow g.$$

Then, as in part (a)

$$\begin{aligned} P_n (B - T) x_n - p (I - T) x_n &= P_n B x_n - p x_n - (1 - p) T x_n = P_n (B - (1 - p) T) x_n - p x_n \\ &= (I - T) g_n \rightarrow (I - T) g \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since $(B - (1 - p)T)$ is P -compact, there exists a subsequence $\{x_{n_i}\}$ and an element x in $D(B)$ such that $x_{n_i} \rightarrow x$ and $P_{n_i} (B - (1 - p)T) x_{n_i} \rightarrow (B - (1 - p)T)x$. This implies that $P_{n_i} A_1 x_{n_i} \rightarrow A_1 x$ as $n_i \rightarrow \infty$, i.e., A_1 is P -compact.

THEOREM 5.6. *Let B be a P -compact mapping of $D(B) \subset X$ into X . Then B has at least one fixed point in $D(B)$ if the conditions of Theorem 5.5 are satisfied for*

(a) *The operator $A = (I - T)^{-1} B (I - T)$, where T is a bounded linear mapping of X into X such that $(I - T)$ is continuously invertible and X_n reduces $(I - T)^{-1}$ for each n .*

(b) *The operator $A_1 = (I - T)^{-1} (B - T)$, where T is a completely continuous operator satisfying the conditions of (a).*

Proof. (a) It follows from Lemma 5.2 and our conditions that $A = (I - T)^{-1} \times B(I - T)$ is P -compact and satisfies the conditions of Theorem 5.5. Hence there exists a ball $B_{r^*}(x_0) \subset D(B) = D(A) = X$ such that

$$A x = (I - T)^{-1} B (I - T) x = x$$

for some x in $B_{r^*}(x_0)$. This implies that $Bz = z$ for $z = (I - T)x$.

(b) As in part (a), there exists a ball $B_r(x_0) \subset D(B) = D(A)$ such that

$$A_1 y = (I - T)^{-1} (B - T) y = y$$

for some y in $B_r(x_0)$. This implies that $By = y$.

REMARK 5.4. In case B is completely continuous and T is also a linear completely continuous operator, Theorem 5.6(b) for the stronger conditions (a) and (b) was established in [11].

REMARK 5.5. It is not hard to see that (with $k = \|(I-T)^{-1}\|$) for the operator $A_1 = (I-T)^{-1}(B-T)$, the conditions (a) and (b) of Theorem 5.5 are implied by either of the following two conditions:

(i) There are $c_1 > 0$, $0 < m_1 < k^{-1}$ and $r > 0$ such that

$$\|(B-T)x - (I-T)x_0\| \leq c_1 \quad \text{for } x \text{ in } S_r(x_0) \quad (\text{or for } x \text{ in } B_r(x_0) \subset D(B)).$$

If $r < c_1 k$ then assume also $\|(B-T)x - (B-T)y\| \leq m_1 \|x - y\|$ for each x in $S_{r_2}(x_0)$ with $r_2 = k(c_1 - m_1 k)/(1 - m_1 k)$ and $y = \alpha x_0 + (1 - \alpha)x \in S_r(x_0)$.

(ii) There are $r > 0$ and $m < k^{-1}$ such that

$$\|(B-T)x - (B-T)y\| \leq m \|x - y\| \quad \text{if } \|x\| > r \text{ and } \|y\| > r.$$

Similarly for $A = (I-T)^{-1}B(I-T)$ a sufficient condition is either of the following conditions:

(j) There are $c > 0$, $0 < m < k^{-1}$ and $r > 0$ such that

$$\|B(I-T)x - (I-T)x_0\| \leq c \quad \text{for } x \text{ in } S_r(x_0).$$

If $ck > r$ then assume also $\|B(I-T)x - B(I-T)y\| \leq m \|x - y\|$ for each x in $S_{r_3}(x_0)$ with $r_3 = k(c - mk)/(1 - mk)$ and $y = \alpha x_0 + (1 - \alpha)x \in S_r(x_0)$.

(jj) There are $r > 0$ and $m < k^{-1}$ such that

$$\|B(I-T)x - B(I-T)y\| \leq m \|x - y\| \quad \text{if } \|x\| > r \text{ and } \|y\| > r.$$

We complete this section with another application of the P -compactness concept to the fixed point theorem for the mappings of the form $A = S + T$, where S is a nonexpansive mapping and T is a strongly continuous mapping of a uniformly convex Banach space X into X . The fixed point theory for such mappings, as well as for somewhat more general (e.g., semicontractive) mappings, has been recently developed by Browder [4] who in his discussion employed the entire machinery of monotone and J -monotone operators. In our discussion below we will not use the theory of J -monotone operators defined below. We should add that, as was shown in [4], $A = S + T$ need not have a fixed point if we assume that T is only completely continuous even if A maps $B_r(0) \subset H$ into $B_r(0)$. Furthermore, it is known that if X is a general Banach space, then a nonexpansive mapping A of $B_r(0)$ into $B_r(0)$ need not have a fixed point. Thus, certain restrictions on A and/or X have to be imposed in order for A to have fixed points.

Let $\mu(r)$ be a continuous strictly increasing real-valued function with $\mu(0) = 0$. A mapping J of X into X^* is called a *duality mapping* with gauge function μ if $\|Jx\| = \mu(\|x\|)$ and $(Jx, x) = \|x\|\mu(\|x\|)$ for each x in X . An operator A of X into X is said to be J -monotone if $(Ax - Ay, J(x - y)) \geq 0$ for all x and y in X . For the discussion of duality mappings and J -monotone operators see [4], [5].

In what follows we shall make use of the following two lemmas which in a somewhat different form were first proved in [22]. For the sake of completeness we prove here these lemmas by using a combination of arguments in [33], [22], [28].

LEMMA 5.3. *Let X be a uniformly convex Banach space having a weakly continuous duality mapping J of X into X^* . Let $\{x_n\}$ be a sequence in X such that $x_n \rightarrow x_0$. Then there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that for any x in X*

$$(5.9) \quad \lim_m \|x_m - x\| \geq \lim_m \|x_m - x_0\|$$

with equality holding in (5.9) if and only if $x = x_0$.

Proof. It is obvious that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $\|x_m - x_0\| \rightarrow d_0$ and $\|x_m - x\| \rightarrow d$ as $m \rightarrow \infty$. Hence, since $x_m - x_0 \rightarrow 0$ implies $J(x_m - x_0) \rightarrow 0$, the definition of J and the passage to the limit in

$$(J(x_m - x_0), x_m - x_0) \leq \|J(x_m - x_0)\| \|x_m - x\| + |(J(x_m - x_0), x - x_0)|$$

imply that $\mu(d_0)d_0 \leq \mu(d_0)d$ from which (5.9) follows. To complete the proof, suppose that $d_0 = d$ in (5.9). Since

$$\|x_m - (tx + (1-t)x_0)\| \leq t\|x_m - x\| + (1-t)\|x_m - x_0\|$$

for any t in $[0, 1]$, it follows from the first part of Lemma 5.3 and our assumption $d_0 = d$ that $\|x_m - x_0\| \rightarrow d_0$, $\|x_m - x\| \rightarrow d_0$, and $\|x_m - (tx + (1-t)x_0)\| \rightarrow d_0$. Hence, by uniform convexity of X , $\|(x_m - x_0) - (x_m - x)\| \rightarrow 0$, i.e., $x = x_0$.

A consequence of Lemma 5.3 is the following useful result.

LEMMA 5.4. *Let X be a uniformly convex Banach space with a weakly continuous duality mapping J of X into X^* . If $A = S + T$ maps a closed convex set $D(A) = C$ of X into X , where S is nonexpansive on C and T strongly continuous, then $(I - A)$ is strongly closed.*

Proof. Let $\{x_n\}$ be a sequence in C so that $x_n \rightarrow x_0$ and let $x_n - Ax_n \rightarrow y_0$ for some y_0 in X . Since C is closed and convex and hence weakly closed, x_0 lies in C and $Tx_n \rightarrow Tx_0$ by strong continuity of T . Hence it suffices to show that $(I - S)x_0 = y_0 + Tx_0$. Let $\{x_m\}$ be a subsequence of $\{x_n\}$ so that $\lim_m \|x_m - x_0\|$ exists. Since $(I - S)x_m \rightarrow y_0 + Tx_0$, there exists a sequence $z_m \rightarrow 0$ such that $Sx_m = x_m - y_0 - Tx_0 + z_m$ and

$$\begin{aligned} \|x_m - x_0\| &\geq \|Sx_m - Sx_0\| = \|x_m - y_0 - Tx_0 - Sx_0 + z_m\| \\ &\geq \|x_m - y_0 - Tx_0 - Sx_0\| - \|z_m\|, \end{aligned}$$

whence, since $z_m \rightarrow 0$, we deduce

$$\lim_m \|x_m - x_0\| \geq \lim_m \|x_m - y_0 - Tx_0 - Sx_0\|.$$

It follows from this and Lemma 5.3 that $x_0 = y_0 - Tx_0 - Sx_0 = y_0 - Ax_0$.

REMARK 5.6. Lemma 5.4 was first proved (for $T=0$) in a more general setting in [4] using the theory of J -monotone operators under the assumption that A is defined on all of X . In a case of a Hilbert space a somewhat different proof is given in [7].

THEOREM 5.7. *Let X be a uniformly convex Banach space with a weakly continuous duality mapping J of X into X^* and with $\|P_n\|=1$ for all n . Let $A=S+T$ be a mapping of X into X , where S is nonexpansive and T is strongly continuous. Suppose that for some bounded closed convex set C with 0 in $\text{int } C$, A satisfies $(\pi_1^<)$ on ∂C . Then there exists a point x_0 in $(C-\partial C)$ such that $Sx_0+Tx_0=x_0$.*

Proof. By Corollary 2.4, $S-I$ is P -compact; furthermore, since a uniformly convex Banach space is reflexive and T is strongly continuous, it follows that T is completely continuous and, therefore, by Lemma 2.1, $S+T-I$ is P -compact. Suppose now that $(A-I)x=\alpha x$ for some x in ∂C . Then $Ax=(1+\alpha)x$ and therefore, since A satisfies condition $(\pi_1^<)$ on ∂C , $\alpha \leq 0$. Hence, by Theorem 4.2, for each $\mu_k > 0$ with $\mu_k \rightarrow 0$ as $k \rightarrow \infty$ there exists an element x_k in $(C-\partial C)$ such that $(A-I)x_k=\mu_k x_k$. But, since $\{x_n\} \subset C$, a closed bounded convex set, and X is reflexive, passing to a subsequence we may assume that $x_k \rightarrow x_0$, where x_0 is some element in C . Thus $x_k \rightarrow x_0$ and $(I-A)x_k=-\mu_k x_k \rightarrow 0$ as $k \rightarrow \infty$. Since, by Lemma 5.4, $(I-A)$ is strongly closed, it follows that $(I-A)x_0=0$ or $Sx_0+Tx_0=x_0$.

REMARK 5.7. As was already noted, Theorem 5.7 was proved in [4] by means of the theory of J -monotone operators for the class of semicontractive mappings U of X into X (which contains mappings $A=S+T$) but under the stronger condition, namely, that U maps C into C which, for $U=S+T$ reduces to the requirement that $Sx+Ty \in C$ for all x and y in C .

6. Iterative and general approximation methods for P -compact mappings.

ITERATIVE METHOD. Extending certain results of [16], [33] and [26], among other results, it was shown in [6] that if X is a uniformly convex Banach space, C a closed convex subset of X and A a contractive (nonexpansive) mapping of C into C with a nonempty set of fixed points in C , then for each given x_0 in C and each fixed λ in $(0, 1)$, the sequence $\{x_{n+1}\}$ determined by the iteration method

$$(6.1) \quad x_{n+1} = \lambda Ax_n + (1-\lambda)x_n \quad (n = 0, 1, 2, \dots)$$

converges strongly to a fixed point of A in C provided A satisfies the condition

$$(\alpha) \quad (I-A) \text{ maps every bounded closed subset of } C \text{ into a closed subset.}$$

It is easy to see that condition (α) is implied by the stronger assumption that A is completely continuous. Furthermore, condition (α) is a slight generalization of the concept of *demicompactness*⁽⁶⁾ of A introduced and studied in [26].

⁽⁶⁾ A is said to be *demicompact* if whenever $\{u_n\}$ is a bounded sequence and $\{u_n - Au_n\}$ is strongly convergent, then $\{u_n\}$ contains a strongly convergent subsequence. It was asserted in [6] that condition (α) is equivalent to the demicompactness. This is clearly not the case for, as was observed in [28], $A=I$ satisfies condition (α) but is not demicompact. The authors of [6] caught this inaccuracy too late for the correction to be inserted in [6]. For details, historical development and recent contributions to the iterative method for the construction of fixed points of contractive type mappings see [7], [28].

To discuss the problem of convergence of the method (6.1) in case A is a P -compact mapping we first derive some preliminary results.

LEMMA 6.1. (a) *Let A be a nonexpansive P_1 -compact mapping of $D(A)=X$ into X . Then A satisfies condition (α) .*

(b) *Suppose additionally that $\|P_n\|=1$ for all n . Then if A is a nonexpansive P_1 -compact mapping of $D(A)=B_r(0)$ into X , then A satisfies condition (α) on $B_r(0)$ (i.e., $(I-A)$ maps every closed subset of $B_r(0)$ into a closed subset of X).*

Proof. Let Q be a bounded closed set in X and let $\{u_k\}$ be a sequence in Q such that $(I-A)u_k \rightarrow v$ as $k \rightarrow \infty$, where v is some element in X . Since the system $(\{X_n\}, \{P_n\})$ is projectionally complete in X , for each integer $k > 0$ there exists an integer $n(k) > k$ such that $\|u_k - w_{n(k)}\| < 1/k$, where we have put $w_{n(k)} = P_{n(k)}u_k$. This and the relation $(I-A)u_k \rightarrow v$ as $k \rightarrow \infty$ imply that

$$\begin{aligned} \|P_{n(k)}Aw_{n(k)} - w_{n(k)} + v\| &\leq \|P_{n(k)}Aw_{n(k)} - P_{n(k)}Au_k\| \\ &\quad + \|P_{n(k)}Au_k - w_{n(k)} + P_{n(k)}v\| + \|v - P_{n(k)}v\|. \end{aligned}$$

Since $\{P_n\}$ is uniformly bounded by $K \geq 1$, A is nonexpansive, and $w_{n(k)} = P_{n(k)}u_k$, it follows from the above inequality and our assumptions that

$$\begin{aligned} \|P_{n(k)}Aw_{n(k)} - w_{n(k)} + v\| &\leq K\|w_{n(k)} - u_k\| + K\|Au_k - u_k + v\| \\ &\quad + \|v - P_{n(k)}v\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Since $\{w_{n(k)} \mid w_{n(k)} \in X_{n(k)}\}$ is bounded and A is P_1 -compact, the preceding relation implies that there exists a subsequence $\{w_{n(j)}\}$ of $\{w_{n(k)}\}$ and an element w such that $w_{n(j)} \rightarrow w$ and $P_{n(j)}Aw_{n(j)} \rightarrow Aw$ as $j \rightarrow \infty$. Consequently, $\|u_j - w\| \leq \|u_j - w_{n(j)}\| + \|w_{n(j)} - w\| \rightarrow 0$ as $j \rightarrow \infty$. Since $\{u_j\}$, being a subsequence of $\{u_k\}$, lies in Q and Q is closed it follows that $w \in Q$ and $Aw - w = v$, i.e., $(I-A)$ satisfies condition (α) .

The validity of assertion (b) follows from the same arguments as above since the condition that $Q \subset B_r(0)$ and that $\|P_n\|=1$ for all n imply that $w_{n(k)} \in B_r \cap X_{n(k)}$ for all $n(k)$. The rest of the proof is the same as that of (a).

LEMMA 6.2. *Let A be a continuous P_1 -compact mapping of a bounded closed convex set C with interior into X and let F be a closed convex set lying in $\text{Int } C$ such that $Ax \neq x$ for all x in $(C-F)$. If we denote by $R(F, \alpha) = \{x \in C \mid d(x, F) \geq \alpha\}$ for some $\alpha > 0$, then for each given $\alpha > 0$ there exists an $\varepsilon = \varepsilon(\alpha) > 0$ such that $\|x - Ax\| > \varepsilon$ for all x in $R(F, \alpha)$.*

Proof. Let us remark that in view of our conditions on F the set $R(F, \alpha)$ is not empty for sufficiently small $\alpha > 0$. First we show that to a given $\alpha > 0$ there exists an $\varepsilon(\alpha) > 0$ and an integer $n_0 > 0$ such that for all $n \geq n_0$

$$(6.2) \quad \|x - P_n Ax\| > \varepsilon(\alpha) \quad \text{for all } x \text{ and } y \text{ in } X_n \cap R(F, \alpha).$$

If this were not the case, then there would exist a number $\alpha_0 > 0$ and a sequence $\{x_m \mid x_m \in X_m \cap C\}$ such that $d(x_m, F) \geq \alpha_0$ and

$$\|x_m - P_m Ax_m\| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Thus, by P_1 -compactness of A , there exists a subsequence $\{x_{m_i}\}$ and an element x in C such that $x_{m_i} \rightarrow x$ and $P_{m_i}Ax_{m_i} \rightarrow Ax$, as $m_i \rightarrow \infty$, and $x - Ax = 0$ contradicting the fact that $d(x_{m_i}, F) \geq \alpha_0$ and $x_{m_i} \rightarrow x$. Hence, (6.2) holds. Furthermore, since $P_m x = x$ for x in X_m and $\|P_n\| \leq K$, it follows from (6.2) that

$$(6.3) \quad \|x - Ax\| \geq \varepsilon(\alpha) = \bar{\varepsilon}(\alpha)/K \quad \text{for all } x \text{ in } X_n \cap R(F, \alpha) \text{ and all } n \geq n_0.$$

Note that (6.2) and (6.3) have been derived without using the continuity of A .

Now let y be an arbitrary element in $R(F, \alpha)$. There are four possible cases: (a) $d(y, F) > \alpha$ and $y \notin \partial C$; (b) $d(y, F) > \alpha$ and $y \in \partial C$; (c) $d(y, F) = \alpha$ and $y \notin \partial C$; (d) $d(y, F) = \alpha$ and $y \in \partial C$.

(a) Since $d(y, F) > \alpha$ and $y \notin \partial C$, there exists a ball $B_\delta(y)$ about y with radius $\delta > 0$ such that $B_\delta(y) \subset C \cap R(F, \alpha)$. Since $\bigcup_n X_n$ is dense in X , we can find a sequence $x_n \in X_n$ such that $x_n \rightarrow y$ as $n \rightarrow \infty$. Hence, there exists an $N > 0$ such that $x_n \in B_\delta(y)$ for all $n \geq N$. Consequently, (6.3) implies that $\|x_n - Ax_n\| > \varepsilon(\alpha)$ for all $n \geq N$. This and the continuity of A implies that $\|y - Ay\| \geq \varepsilon(\alpha)$.

Cases (b), (c), and (d) can be handled by one technique. Suppose for definiteness that $d(y, F) = \alpha$ and $y \notin \partial C$. In virtue of our conditions on F , we can find a sequence $\{y_n\}$ in $R(F, \alpha)$ such that $d(y_n, F) = \alpha_n > \alpha$ and $y_n \rightarrow y$ as $n \rightarrow \infty$. Then for each n , by case (a), $\|y_n - Ay_n\| > \varepsilon(\alpha)$ and hence again, by continuity of A , we have $\|y - Ay\| \geq \varepsilon(\alpha)$ for all y in $R(F, \alpha)$.

REMARK 6.1. An analogous lemma for completely continuous operators is known (e.g., see [17]) to play a very important role in the study of completely continuous operator equations. It is, therefore, expected that Lemma 6.2 will also be of considerable value in the study of P -compact operator equations. Its importance in our investigation will be seen below.

THEOREM 6.1. *Let X be a uniformly convex Banach space and A a nonexpansive P_1 -compact mapping of $D(A) \subset X$ into X .*

(a) *If $D(A) = X$ and, for some bounded closed convex set C with an interior, A maps C into C . Then for any $x_0 \in C$ and any fixed λ in $(0, 1)$ the sequence x_{n+1} determined by the iteration method (6.1) converges strongly to a fixed point of A in C .*

(b) *If $D(A) = B_r(0)$ and A maps $B_r(0)$ into $B_r(0)$, then the same assertion holds provided $\|P_n\| = 1$.*

(c) *If $D(A) = C$, where C is as in (a), and A is a map of C into C with no fixed points on ∂C , then the assertion of (a) remains valid.*

Proof. (a) In virtue of Lemma 6.1(a) and the result in [6] quoted above, it suffices to prove that A has a fixed point in C . But, under our conditions on A and C , Corollary 4.2 implies that A has fixed points in C .

(b) The proof of Theorem 6.1(b) follows from the same arguments and Lemma 6.1(b).

(c) To prove assertion (c) first note that, by Corollary 4.2, the set F of fixed points of A in C is nonempty and, in fact, F is a convex closed set (see [33]). Since,

by assumption, A has no fixed points on the boundary ∂C , $F \subset \text{Int } C$. Now, it is easy to show (see [16]) that, for the sequence $\{x_{n+1}\}$ defined by (6.1), $d(x_{n+1}, F) \leq d(x_n, F)$ for each n . Hence, $\{d(x_n, F)\}$ is a monotonically decreasing sequence of real numbers which is bounded below by zero. We claim that $d(x_n, F) \rightarrow 0$ as $n \rightarrow \infty$. Suppose, to the contrary, that $d(x_n, F) \rightarrow \alpha_0 > 0$ as $n \rightarrow \infty$. By monotonicity, $d(x_n, F) > \alpha_0$ for each n . Hence, by Lemma 6.2, there exists $\varepsilon_0 = \varepsilon(\alpha_0) > 0$ such that $\|x_n - Ax_n\| \geq \varepsilon_0$. Letting $v_n = x_n - u$ and $w_n = Ax_n - u$ with u in F , we see that, since $\|x_n - Ax_n\| \leq \|x_{n-1} - Ax_{n-1}\|$, $\|w_n - v_n\| > \varepsilon_0$ for $n \geq 0$ and

$$\|x_{n+1} - u\| = \|\lambda(Ax_n - Au) + (1-\lambda)(x_n - u)\| = \|\lambda w_n + (1-\lambda)v_n\|.$$

Hence, by Theorem 1 in [33], there exists a $\delta_0 = \delta(\varepsilon_0, \lambda, C)$ so that

$$\|x_{n+1} - u\| < (1 - \delta_0)\|x_n - u\| \quad \text{for each } n.$$

Consequently, $\|x_n - Ax_n\| < 2(1 - \delta_0)\|x_{n-1} - u\|$. The last two relations imply that $\|x_n - Ax_n\| \rightarrow 0$ as $n \rightarrow \infty$. Hence, the contradiction is reached and, therefore, $d(x_n, F) \rightarrow 0$ as $n \rightarrow \infty$. From the arguments of Krasnoselsky [16] it then follows that $\{x_{n+1}\}$ is a Cauchy sequence. Since C is closed and $\{x_{n+1}\} \in C$, there exists an x in C such that $x_n \rightarrow x$ as $n \rightarrow \infty$. The continuity of A then implies that x is a fixed point of A .

NOTE. We note, for example, that if X is such that $\|P_n\| = 1$ then Theorem 6.1(a) is applicable to nonexpansive operators of the form $A = S + T$, where S is Lipschitzian on $D(A) = X$ with $L < 1$ and T is completely continuous; if, however, $A = S + T$ is defined only on $D(A) = B_r(0)$ and $L < 1/2$, then Theorem 6.1(b) is applicable. Note that in both cases A is P_1 -compact but not completely continuous so that the results in [16], [33] are not applicable.

APPROXIMATION METHODS. Let A be a bounded continuous P -compact mapping of $D(A) = X$ into X . In this section we extend certain results of [24] concerning the projection method for the approximate solution of the equation

$$(6.4) \quad Ax = x$$

to the approximations of a more general type. Our results will in particular include the recent results of Vainikko [39] for completely continuous mappings.

For the approximate solution of equation (6.4) we take the solutions of the equations

$$(6.5) \quad A_n x = x$$

where the A_n are bounded P -compact mappings of X into X which in some sense (to be defined below) approximate A .

THEOREM 6.2. *Let A and A_n be as above. Suppose that A satisfies condition $(\pi_1^<)$ on ∂B_r , for some $r > 0$. Let A have at most one fixed point in $(B_r - \partial B_r)$. Suppose further that*

$$(6.6) \quad \tau_n = \sup_{x \in B_r} \|A_n x - Ax\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then A has a (unique) fixed point x_0 in $(B_r - \partial B_r)$ and there exists an integer $N > 0$ such that for each $n \geq N$ equation (6.5) has a solution x_n in $(B_r - \partial B_r)$ so that $x_n \rightarrow x_0$ as $n \rightarrow \infty$.

If, in addition, we assume that $A_n(x)$ are uniformly in n Fréchet-differentiable at $x = x_0$ and that, for sufficiently large n ,

$$\|x - A'_n(x_0)x\| \geq \beta \|x\| \quad \text{for each } x \text{ in } B_r$$

and some constant $\beta > 0$, then the estimate of the error $\|x_n - x_0\|$ is given by

$$(6.7) \quad \|x_n - x_0\| \leq \frac{1}{\beta - \theta_n} \|A_n x_0 - A x_0\|,$$

where $\theta_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. First note that in virtue of our conditions on A , Theorem A implies that A has a fixed point x_0 in $(B_r - \partial B_r)$ which, by assumption, is unique. Now by Lemma 6.2, there exists an ε such that for all x in ∂B_r , $\|x - Ax\| \geq \varepsilon$. Since $\tau_n \rightarrow 0$ as $n \rightarrow \infty$ we can find an N such that for all $n \geq N$, $\tau_n \leq \varepsilon/2$. Then, for all x in ∂B_r and all $n \geq N$

$$\|x - A_n x\| \geq \|x - Ax\| - \|A_n x - Ax\| \geq \varepsilon/2.$$

Thus, for $n \geq N$, A_n have no fixed point on ∂B_r , and so by results of Tucker [36], for all $n \geq N$, there exist x_n in $(B_r - \partial B_r)$ such that $A_n x_n - x_n = 0$.

Now let $\varepsilon > 0$ be arbitrary and consider the set F_ε of all x in B_r such that $\|x - x_0\| \geq \varepsilon$. Since x_0 is the unique fixed point of A in B_r , by Lemma 6.2, there exists $\alpha(\varepsilon) > 0$ such that $\|x - Ax\| \geq \alpha(\varepsilon)$ for x in F_ε . As above, for sufficiently large n and all x in F_ε , we have

$$\|x - A_n x\| \geq \|x - Ax\| - \|Ax - A_n x\| \geq \alpha(\varepsilon) - \alpha(\varepsilon)/2 = \alpha(\varepsilon)/2 > 0.$$

Thus, for sufficiently large n , the approximate equations (6.5) cannot have solutions x_n outside an ε -neighborhood of x_0 . Since ε is arbitrary, $\|x_n - x_0\| \rightarrow 0$ as $n \rightarrow \infty$.

To prove the second part of Theorem 6.2 let us first observe that since A'_n is a Fréchet derivative of $A_n(x)$ at $x = x_0$ it follows that

$$A_n(x_n) = A_n[x_0 + (x_n - x_0)] = A_n x_0 + A'_n(x_n - x_0) + w_n(x_0; x_n - x_0).$$

Furthermore, since $A_n(x)$ are uniformly in n Fréchet-differentiable and, by the first part of our Theorem, $x_n \rightarrow x_0$ as $n \rightarrow \infty$, it follows that

$$(6.8) \quad \theta_n = \frac{\|w_n(x_0; x_n - x_0)\|}{\|x_n - x_0\|} = \frac{\|A_n x_n - A_n x_0 - A'_n(x_n - x_0)\|}{\|x_n - x_0\|} \rightarrow 0,$$

as $n \rightarrow \infty$. Using the above relation and the fact that $Ax_0 = x_0$ and $A_n x_n = x_n$ we obtain

$$\begin{aligned} x_n - x_0 &= A_n x_n - Ax_0 = A_n x_n - A_n x_0 + A_n x_0 - Ax_0 \\ &= A'_n(x_0)(x_n - x_0) + w_n(x_0; x_n - x_0) + A_n x_0 - Ax_0, \end{aligned}$$

whence we get the looked-for equality

$$(6.9) \quad \{I - A'_n(x_0)\}(x_n - x_0) = A_n x_0 - Ax_0 + w_n(x_0; x_n - x_0).$$

In view of (6.6) and the equality (6.8), we derive from (6.9) the following

$$(6.10) \quad \beta \|x_n - x_0\| \leq \|A_n x_0 - Ax_0\| + \theta_n \|x_n - x_0\|.$$

Since, by (6.8), $\theta_n \rightarrow 0$ as $n \rightarrow \infty$, we see that $\beta - \theta_n > 0$ for sufficiently large n . Hence the error estimate follows from (6.10). This completes the proof of Theorem 6.2.

REMARK 6.2. In case $A_n = P_n A$, Theorem 6.2 reduces to Theorem 7 in [24] for projectional methods under somewhat weaker conditions.

REMARK 6.3. If, in Theorem 6.2, we replace $B_r(0)$ by $B_r(x_0)$, then by employing a translation process similar to that used in Theorem 4.4, we obtain the corresponding result of Vainikko [39] when A and A_n are completely continuous.

REMARK 6.4. Finally, we remark that if additionally we assume that $A_n(x)$ are uniformly in n continuous Fréchet-differentiable at $x = x_0$, then by arguments analogous to those in [39] one can show that for sufficiently large n , the approximate equations are uniquely solvable.

It follows from Theorem 6.2 that if x_0 is a fixed point of a nonlinear mapping A , it is important to know under what conditions x_0 is an isolated fixed point of A . In this direction we prove the following theorem.

THEOREM 6.3. *Let x_0 be a fixed point of a mapping A . If $A(x)$ has a P -compact Fréchet derivative $A'(x_0)$ at $x = x_0$ such that $x = 0$ is the only solution of the equation $x - A'(x_0)x = 0$, then x_0 is an isolated fixed point of A .*

Proof. Since $A'(x_0)$ is a linear P -compact mapping of X into X and $\lambda = 1$ is not an eigenvalue of $A'(x_0)$, it follows from Theorem 3.1 and Lemma 6.2 that there exists a constant $c > 0$ such that $\|A'(x_0)x - x\| \geq c\|x\|$ for all x in X . Since, by definition of $A'(x_0)$, for each h in X , $A(x_0 + h) - A(x_0) = A'(x_0)h + w(x_0; h)$, where $A'(x_0)$ is a linear mapping of X into X and

$$(6.11) \quad \lim_{\|h\| \rightarrow 0} \frac{\|w(x_0; h)\|}{\|h\|} = 0,$$

it follows from (6.11) that there exists a number $r_0 > 0$ such that $\|x\| < r_0$ implies the inequality

$$\|w(x_0; x)\| = \|A(x_0 + x) - A(x_0) - A'(x_0)x\| \leq \frac{c}{2} \|x\|.$$

Thus, on the spheres $S_r(x_0)$ of radii $r \leq r_0$, A has no fixed points since

$$\|A(x_0 + x) - (x_0 + x)\| \geq \|A'(x_0)x - x\| - \|w(x_0; x)\| \geq \left(c - \frac{c}{2}\right)\|x\| = \frac{c}{2}\|x\|.$$

This shows that x_0 is an isolated fixed point of A .

REMARK 6.5. If $X = H$, x_0 is a fixed point of a mapping A of $D(A)$ into H and A has a Fréchet derivative $A'(x_0)$ at $x = x_0$ then, for example, by Corollary 2.1 the mapping $A'(x_0)$ is P -compact if $(A'(x_0)h, h) \leq 0$ for all h in H . Note that $A'(x_0)$ need not be symmetric.

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